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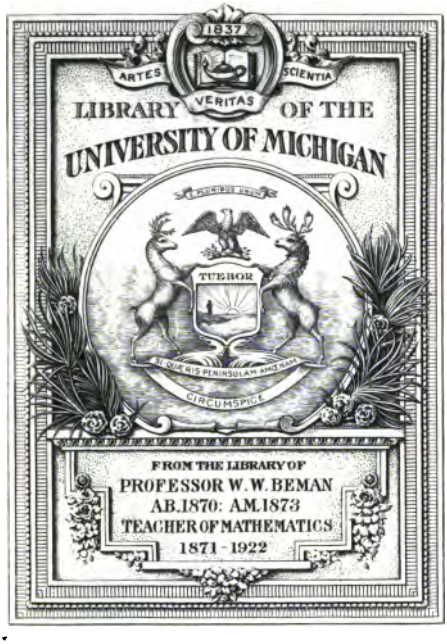
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WITH MANY

PAPERS AND SOLUTIONS

IN ADDITION TO THOSE

PUBLISHED IN THE "EDUCATIONAL TIMES,"

AND FIVE

APPENDICES.

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2081. (The late Professor Rutherford, LL.D.) — A triangle A'B'C' has its angles respectively complementary to the half angles of the triangle ABC, and its side B'C' equal to BC; prove that

$$\Delta A'B'C' : \Delta ABC = \sin \tfrac{1}{2}A : 2 \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C. \text{ ..... } 173$$

2109. (Prof. Wolstenholme, M.A., Sc.D.)—If A_1, A_2, A_3, A_4, A_5 be five fixed points in space of three dimensions, P any other point, V_1, V_2, ... the volumes of the tetrahedra $A_2A_3A_4A_5$, &c., taken of such signs that

$$V_1 + V_2 + V_3 + V_4 + V_5 = 0,$$

prove that (1) $V_1 . PA_1^2 + V_2 . PA_2^2 + V_3 . PA_3^2 + V_4 . PA_4^2 + V_5 . PA_5^2$ will be independent of the position of P; and (2) if C_1, C_2, C_3, C_4, C_5 be the centres of the spheres circumscribing the tetrahedra $A_2A_3A_4A_5$, &c., r_1, r_2, r_3, r_4, r_5 the radii of these spheres,

$$\frac{1}{C_1A_1^2 - r_1^2} + \frac{1}{C_2A_2^2 - r_2^2} + \frac{1}{C_3A_3^2 - r_3^2} + \frac{1}{C_4A_4^2 - r_4^2} + \frac{1}{C_5A_5^2 - r_5^2} = 0,$$

$$V_1(C_1A_1^2 - r_1^2) = V_2(C_2A_2^2 - r_2^2) = \ldots = V_1 . PA_1^2 + \ldots .$$

[The corresponding result for two dimensions, at least the equation $\frac{1}{C_1A_1^2 - r_1^2} + \ldots = 0$, was set in the Mathematical Tripos Examination for 1886; the proof is precisely the same for both, and is a rather nice example of the use of determinants.] 177

2814. (The late Matthew Collins, B.A.)—Can the common difference of three rational square integers in A. P. be ever equal to 17?... 192

2903. (Professor Sylvester, F.R.S.)—If i be any positive integer, and $(\Delta-3)^i u_x = \frac{\Pi(2x)}{(\Pi x)^2}$, show that

$$u_x = (1,x)^{i-1}\,4^x + \frac{\Pi i \,.\, \Pi(2x+2i)}{\Pi 2i \,.\, \Pi x \,.\, \Pi(x+i)}.$$

[This theorem virtually contains a solution of the partial difference-equation (not noticed by the late Prof. Boole), $u_{x,y} = u_{x-1,y} + u_{x,y-1}$.] 85

2906. (Professor Sylvester, F.R.S.)—Prove that (1) the continued fraction $1+\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r+\ldots}}}$, where p, q, r are the terms of the natural harmonic progression 1, $\frac{1}{2}$, $\frac{1}{3}$, ... is equal to $\frac{1}{2}\pi$; and (2), in employing the successive convergents to the above continued fraction, explain the advantage of leaving off with a numerator 1 instead of with its denominator. 101

2912. (The Editor.)—Find three rational numbers such that, if their squares be diminished by 2, 3, 5 respectively, the three remainders shall be rational squares 192

2926. (Professor Sylvester, F.R.S.)—Let f, g, h, k be any four given unequal quantities, and m, n, p, q four others (to be found) such that the arc of the curve

$$y = f\log(x^2-m^2) + g\log(x^2-n^2) + h\log(x^2-p^2) + k\log(x^2-q^2)$$

is equal to $x + f\log\frac{x+m}{x-m} + g\log\frac{x+n}{x-n} + h\log\frac{x+p}{x-p} + k\log\frac{x+q}{x-q}$.

Prove that, in general, there are nine systems of values of m, n, p, q which will satisfy the above condition. 157

3228. (Professor Wolstenholme, M.A., Sc.D.)—1. In a tetrahedron ABCD, if AB be perpendicular to CD, and AC to BD, so also will AD be perpendicular to BC. Such a tetrahedron may be called rectangular, and we shall have $BC^2+AD^2 = CA^2+BD^2 = AB^2+CD^2$.

2. In a rectangular tetrahedron, the perpendiculars from the angular points on the opposite faces meet in a point, the centre of perpendiculars. Such a point does not exist for any other tetrahedron.

3. The three straight lines, each intersecting at right angles a pair of opposite edges, meet in the centre of perpendiculars.

4. There exists a polar sphere to any rectangular tetrahedron, that is, each angular point is the pole of the opposite face with respect to a sphere; the centre of this sphere is the centre of perpendiculars. The sphere will not be possible unless all the plane angles, containing one of the solid angles, are obtuse.

5. One sphere can be drawn through the middle points of the six edges, and through the feet of the shortest distances between opposite edges, and the centre of this sphere is the centre of inertia of the tetrahedron, which bisects the distance between the centre of the circumscribed sphere and the centre of perpendiculars. (First twelve-points sphere.)

6. The sphere which passes through the centres of inertia of the faces will also pass through their centres of perpendiculars; and these points in any face will be the ends of a diameter of the circle in which the face cuts the sphere. This sphere also trisects the part of any perpendicular of the tetrahedron intercepted between the centre of perpendiculars and the corresponding angular point. (Second twelve-points sphere.)

7. These spheres and the circumscribed sphere have a common radical plane, and to this system of spheres belongs also the sphere of which the centre of inertia and the centre of perpendiculars are ends of a diameter.

8. If R, ρ be the radii of the circumscribed sphere and the polar sphere, and δ the distance between their centres, $\delta^2 = R^2 + 3\rho^2$; and the radius of the first twelve-points sphere is $\frac{1}{2}(R^2-\rho^2)^{\frac{1}{2}}$, that of the second is $\frac{1}{2}R$, and that of the sphere in (7) is $\frac{1}{4}(R^2+3\rho^2)^{\frac{1}{2}}$. The radius of the common section of all these spheres is $\frac{\rho}{\delta}(R^2-\rho^2)^{\frac{1}{2}}$.

9. If R, ρ' be the radii of the circumscribed and first twelve-points spheres, and δ' the distance between their centres, $\delta'^2 = R^2 - 3\rho'^2$.

10. The centre of similarity of the circumscribed and second twelve-points sphere is the centre of perpendiculars of the tetrahedron; whence, if ALA'a be drawn through the centres of perpendiculars to meet the opposite face in A' and the circumscribed sphere in a, $La = 3LA'$..... 167

3836. (R. Tucker, M.A.)—The digits of a number of two figures are multiplied together, and if the resulting number consists of two figures these are also multiplied together, and the process continued until there is a single digit only; find the respective chances of this ultimate digit being (1) 5, or (2) 8, or (3) a square number; also (4) find the same of a number of three digits..... 192

4129. (The Editor.) — If three given weights (considered as heavy material points) are attached to the surface of a sphere, find (1) the position of equilibrium of the sphere when resting on a horizontal plane; and (2) give also the result in the particular case in which the weights are arranged in a great circle. 76

4208. (H. S. Monck, M.A.)—Three lines AL, BM, CN are drawn from the vertices of a triangle ABC through a point O to the opposite sides; prove that $2\triangle ABC : \triangle LMN = AO \cdot BO \cdot CO : LO \cdot MO \cdot NO$..... 165

5806. (W. J. C. Sharp, M.A.)—Find (1) the law of distribution of mass such that the catenary (under the action of gravity) may be a given curve, for example, the portion of an ellipse cut off by its horizontal major axis; and (2) supposing the mass supplied by a fringe of uniform separate threads, find the locus of the ends both generally and in the particular case..... 160

6271. (W. J. C. Sharp, M.A.) — Of all the lines whose satellites, with respect to a nodal cubic, pass through the node, prove that each passes through one of four fixed points on the harmonic polar of the real inflexion..... 160

6306. (Professor Sylvester, F.R.S.)—Given the system of equations

$$x_1(x_1+x_2)(x_1+x_3)\dots(x_1+x_n) = C_1(x_1-x_2)(x_1-x_3)\dots(x_1-x_n),$$
$$x_2(x_2+x_1)(x_2+x_3)\dots(x_2+x_n) = C_2(x_2-x_1)(x_2-x_3)\dots(x_2-x_n),$$
$$\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots$$
$$x_n(x_n+x_1)(x_n+x_2)\dots(x_n+x_{n-1}) = C_n(x_n-x_1)(x_n-x_2)\dots(x_n-x_{n-1});$$

prove that the number of solutions cannot exceed $1^2 \cdot 3^2 \cdot 5 \dots (n-1)^2$ or $1^2 \cdot 3^2 \cdot 5^2 \dots (n-2)^2 n$, according as n is even or odd. 157

6371. (W. J. C. Sharp, M.A.) — If a uniform wire, of such a nature that the resistance to bending varies as the curvature, be suspended, without torsion or extension of the mean fibre, from two points; show that it will hang in a catenary identical with that in which a uniform

string of the same length would hang, but that the tension along the wire will differ by a constant quantity from that along the string. 161

6386. (W. S. McCay, M.A.) — If two spherics cut orthogonally, prove that (1) an indefinite number of pairs of points can be found on their common diameter, such that either point has the same polar to one of the spherics as the other has to the other, and (2) the distance between such a pair of points subtends a right angle at any intersection of the spherics. 167

6392. (E. B. Elliott, M.A.) — A_1, A_2, A_3, A_4 are the vertices of a tetrahedron, P is the point whose volume coordinates with regard to it (*i.e.*, the ratios $PA_2A_3A_4 : A_1A_2A_3A_4$, &c.) are x_1, x_2, x_3, x_4, and O is any other point. Prove that, a_{12} being the length of the edge A_1A_2,

$$OP^2 = x_1OA_1^2 + x_2OA_2^2 + x_3OA_3^2 + x_4OA_4^2 - \Sigma(a_{12}^2 x_1 x_2). \quad \ldots\ldots 176$$

6899. (W. S. B. Woolhouse, F.R.A.S.) [See Corrigenda, p. xxxvi.]— Any straight line passing through the centroid being taken as an axis in a thin lamina having a quadrilateral boundary, let y_1, y_2, y_3, y_4, with relative algebraic signs, denote perpendiculars drawn thereon from the four corners; then, if k be the radius of gyration of the lamina round the said axis, show that $6k^2 = -y_1y_2 - y_1y_3 - y_1y_4 - y_2y_3 - y_3y_4 - y_3y_4$.

[It is also remarkable that the same formula holds good when the axis passes through the centre of the four corners.] 72

6903. (W. J. C. Sharp, M.A.) — Find the law of density, varying as the depth, that the centre of pressure of a semicircle, just immersed with its diameter horizontal, may divide the vertical radius in the ratio of $m : n$. 174

6987. (B. Reynolds, M.A.) — Find (1) in how many essentially different ways eight queens can be placed on an ordinary chess-board, so that no two can take each other; and show (2) that seven queens can be placed on a seven-square board in seven ways, so as not to be able to take each other. 26

7189. (Professor Sylvester, F.R.S.)—Sum the series

$$1 + (x-i) + \frac{(x-2i)(x-2i-1)}{1.2} + \frac{(x-3i)(x-3i-1)(a-3i-2)}{1.2.3} + \ldots\ldots 100$$

7335. (W. J. C. Sharp, M.A.) — If O be the circum-centre of the triangle ABC, and AO, BO, CO be produced to meet the opposite sides in D, E, F, the circle in D′, E′, F′ respectively; prove that

$$\frac{DD'}{AD} + \frac{EE'}{BE} + \frac{FF'}{CF} = 1, \quad BD \cdot DC + OD^2 = AE \cdot EC + OE^2 = AF \cdot FB + OF^2.$$

..... 165

7341. (A. Martin, B.A.)—Solve the equations

$$yz(y+z-x) = a, \quad xz(z+x-y) = b, \quad xy(x+y-z) = c. \ldots\ldots 106$$

7488. (Professor Hudson, M.A.)—If O be the circumcentre of ABC, and forces act along OA, OB, OC proportional to BC, CA, AB; prove that their resultant passes through the in-centre. 164

7533. (J. J. Walker, M.A., F.R.S.)—Prove that the common centroid of the surface-mass of the four faces of a tetrahedron is the centre of the sphere inscribed in that determined by the four centroids of the faces; and hence prove the obvious analogue in tri-dimensional space of Professor Hudson's Question 7488—which is true in any position of the point O, for forces proportional to OA sin A, OB sin B, OC sin C. 164

7660. (R. Knowles, B.A., L.C.P.)—From the angular points of a triangle ABC, lines are drawn through the circum-centre to meet the opposite sides in D, E, F, respectively; prove that

$$\frac{1}{AD}+\frac{1}{BE}+\frac{1}{CF}=\frac{2}{R}. \quad \ldots\ldots\ldots\ldots 165$$

7740. (Professor Sylvester, F.R.S.)—If $px^2+qx+r=0$ be an equation in matrices of the 2nd order, or (if any one pleases to say so) in quaternions, it follows from Hamilton's investigations (see *Lectures on Quaternions*, pp. 631, 632) that this equation has in general 6 roots, reducible to 4 if a certain single ordinary equation is satisfied by the elements of the matrices p, q, r. Show that, if this condition is satisfied for any given system of matrices p, q, r, it will also be satisfied by any three homogeneous linear functions of p, q, r.

[Professor Sylvester recommends the student to prove that the condition referred to is found by equating to zero the discriminant in respect to u, v, w of the determinant to $up+vq+wr$, from which its invariability when p, q, r undergo linear substitution is an immediate consequence. It will, of course, be understood that if

$$p=\begin{matrix}a, & b\\ c, & d\end{matrix},\quad q=\begin{matrix}a', & b'\\ c', & d'\end{matrix},\text{ and }r=\begin{matrix}a'', & b''\\ c'', & d''\end{matrix},\ up+vq+wr \text{ means}$$

$au+a'v+a''w,\quad bu+b'v+b''w,\quad cu+c'v+c''w,\quad du+d'v+d''w.]$ ... 23

7822. (E. Buck, M.A.)—If R be the radius of a circle round a quadrilateral whose sides are a, b, c, d, and $2s\equiv a+b+c+d$, prove that

$$(4R)^2=\left[(ab+cd)(ac+bd)(ad+bc)\right]/\left[(s-a)(s-b)(s-c)(s-d)\right] \ldots. 148$$

7889. (Professor Sylvester, F.R.S.) — If by the developant of ω^2 matrices of order ω be understood in general the determinant to the square matrix formed by *developing* each of them into a line of ω^2 terms, and under- or over-writing these lines in regular order to one another, and if by the quantic associated to i such matrices, say $m_1, m_2, \ldots m_i$ be understood the quantic of order ω, in $x_1, x_2, \ldots x_i$, which is the determinant to the matrix $x_1m_1+x_2m_2+\ldots+x_im_i$; prove that the invariant of the quantic associated to four matrices of the second order is (at all events to a numerical multiplier *près*) the square of their developant. 37

7902. (R. Rawson, F.R.A.S., &c.)—Show that the general integral of

$$A_2^{\frac{1}{2}}\left(1+B_2x^2+\frac{B_2^2A_4}{A_2^2}x^4\right)^{\frac{1}{2}}.\frac{dy}{dx}+B_2^{\frac{1}{2}}(1+A_2y^2+A_4y^4)^{\frac{1}{2}}=0$$

is $[CA_2+B_2A_4x^2]y^2+2[(A_2B_2)^{\frac{1}{2}}(A_4+C^2-CA_2)^{\frac{1}{2}}]xy+A_2+CB_2x^2=0.$

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7954. (W. J. C. Sharp, M.A.) — In a triangle ABC, if p_1 be the perpendicular from A upon BC, r the in-radius and r_1 the ex-radius to BC, show that (1) $\frac{1}{r}-\frac{1}{r_1}=\frac{2}{p_1}$; (2) the same equation holds if p_1 be the perpendicular from the vertex A of a tetrahedron upon the opposite face, and r the radius of the inscribed sphere, and r_1 that of the sphere touching BCD and the other faces produced. 166

7980. (W. J. C. Sharp, M.A.) — If AA′, BB′, &c. be concurrent lines drawn from the vertices of any tetrahedron ABCD to meet the opposite faces in A′, B′, &c., show that

triangle A′BC : triangle A′DB = triangle B′AC : triangle B′DA,
triangle A′BD : triangle A′CD = triangle D′BA : triangle D′AC, &c.

..... 166

7990. (Professor Hanumanta Rau, B.A.)—Find the relation between the positive integers X, n, Y, such that the number $X^2 + nY^2$ may be split up into factors of the form $x^2 + ny^2$. 193

8153. (R. Knowles, B.A., L.C.P.)—A diameter through the pole of CD, a chord of a parabola, meets the curve in P; prove that the area of the triangle PCD is twice that formed by joining the poles of its sides. 88

8019. (W. J. C. Sharp, M.A.)—If the weights of three bodies whose centres are at the vertices of a triangle are connected by a homogeneous relation of the n^{th} order, prove that (1) the locus of the centroid of the system is a plane n-ic curve; and (2), if those of four bodies, whose centroids occupy the vertices of a tetrahedron, are connected by such a relation, the locus is an n-ic surface. 57

8042. (Professor Sylvester, F.R.S.) — Let A, B, C, D be the perpendiculars upon a plane from the points a, b, c, d, the angles of a pyramid whose volume is P. Required (1) to prove that

$$\Sigma (ab)^4 (C-D)^2 - 2\Sigma (ab)^2 (ac)^2 (D-B)(D-C) + 2\Sigma (ab)^2 (cd)^2 (A-C)(A-D) + (B-C)(B-D) = -144P^2.$$

Show also (2) how to find the Constant Homogeneous Quadratic Function of the five perpendiculars from five points in space of four dimensions upon any hyper-plane drawn thereon. 164

8078. (Professor Sylvester, F.R.S.) — If in a system of quadruplanar coordinates, for which $x+y+z+t$ expresses the plane at infinity, $abcd$ is the pyramid of reference; show that (1) $\Sigma (ab)^2 xy$ is the sphere which circumscribes it; and hence, (2) if ξ, η, θ, ζ are the perpendicular distances of a, b, c, d from any variable plane, the following determinant is a constant, and find its value :—

.	ξ	η	θ	ζ	.
ξ	.	$(ab)^2$	$(ac)^2$	$(ad)^2$	1
η	$(ba)^2$	.	$(bc)^2$	$(bd)^2$	1
θ	$(ca)^2$	$(cb)^2$	.	$(cd)^2$	1
ζ	$(da)^2$	$(db)^2$	$(dc)^2$	.	1
.	1	1	1	1	.

..... 164

8098. (W. J. C. Sharp, M.A.) — Show that, if p be any prime and n an integer prime to p, $x^{n(p-1)} + x^{n(p-2)} + x^{n(p-3)} + \ldots + 1$ is divisible by $x^{p-1} + x^{p-2} + x^{p-3} + \ldots + 1$; and, if n be even, also by

$$x^{p-1} - x^{p-2} + x^{p-3} + \ldots + (-1)^{p-1};$$

and that, when n is odd, this last quantity divides

$$x^{n(p-1)} - x^{n(p-2)} + x^{n(p-3)} + \ldots + (-1)^{p-1}. \ldots\ldots 108$$

8140. (H. J. Read, B.A.)—Show that the area of a quadrilateral is half of the value of the determinant

$$\begin{vmatrix} x_1 & y_1 & 10 \\ x_2 & y_2 & 11 \\ x_3 & y_3 & 10 \\ x_4 & y_4 & 11 \end{vmatrix} . \ldots\ldots 96$$

8163. (Professor Sâradâranjan Rây, M.A. Suggested by Question 8008.)—A conic C cuts two given conics A and B. If two conics be drawn, one through the common points of A and C and the other through

those of B and C; then they cut at four points that lie on the same conic with the common points of A and B. 62

8201. (R. Tucker, M.A.)—Six numbers of three figures are formed by different arrangements of three unequal digits; find the digits when the numbers divided by 7 have only three different remainders. 193

8216. (Professor Sylvester, F.R.S.)—Let x, y, z, t be the distances of any point P, and x', y', z', t' of P', from the faces of a fundamental pyramid divided respectively by the distances of the vertices a, b, c, d of the pyramid from those faces; prove that $PP'^2 = \Sigma (ab)^2 (x-x')(y'-y)$. [Of course a precisely similar formula applies to distances on a plane or in space of any number of dimensions.] 164

8242. (Professor Sylvester, F.R.S.) — If by a Simplicissimum of the nth order be understood a figure in space of n dimensions formed by the indefinite protraction of the series of which a linear segment, a triangle, and a pyramid are the three first terms, prove that (1) when each edge is unity, the squared content is $\frac{(n+1)}{2^n (1 \cdot 2 \cdot 3 \ldots n)^2}$; and hence (2) deduce that the Cayleyan Persymmetrical Invertebrate Determinant of the squared edges by which such squared content is imaged must be diminished in the ratio of negative unity to $(-2)^n (1 \cdot 2 \cdot 3 \ldots n)^2$, in order that it may represent its absolute value. *Ex. gr.*, the determinant

$$\begin{vmatrix} . & (ab)^2 & (ac)^2 & 1 \\ (ba)^2 & . & (bc)^2 & 1 \\ (ca)^2 & (cb)^2 & . & 1 \\ 1 & 1 & 1 & . \end{vmatrix}$$

images the squared contents of a triangle whose edges are (ab), (ac), (bc), for the triangle vanishes when this determinant vanishes, but the actual value of the squared content is this determinant diminished in the ratio of negative unity to $2^2 (1 \cdot 2)^2$, *i.e.*, multiplied by $-\frac{1}{16} = (2)$. 164

8279. (Professor Wolstenholme, Sc.D.) — A common tangent to two conics S, S' touches them in A, A', and through a fixed point H on this common tangent is drawn any straight line meeting the two conics in P, Q; P', Q', respectively: prove that the points of intersection of AP, AQ with A'P', A'Q' lie on a fixed conic passing through the four common points of S, S'. 54

8293. (Asparagus.) — From a point O on an ellipse (axes $2a$, $2b$) are drawn three chords OP, OQ, OR normal to the ellipse at P, Q, R; prove that $OP \cdot OQ \cdot OR = \frac{a^2 b^2}{a^2 - b^2} \times$ diameter of curvature at O. ... 137

8294. (D. Edwardes.)—Prove that the equation

$$(h^2 - ab)(ax^2 + 2hxy + by^2 + 2gx + 2fy) - af^2 - bg^2 + 2fgh = 0$$

represents a pair of straight lines, and show how they are connected with the conic given by the general equation. 194

8297. (Piers C. Ward, B.A.)—Prove that, if

$$y^2 + yz + z^2 = a, \quad z^2 + zx + x^2 = b, \quad x^2 + xy + y^2 = c,$$

then

$$(a-b)(4b-3x^2)^{\frac{1}{2}} \pm (a-c)(4c-3x^2)^{\frac{1}{2}} = 3(b-c)x,$$
$$(b-c)(4c-3y^2)^{\frac{1}{2}} \pm (b-a)(4a-3y^2)^{\frac{1}{2}} = 3(c-a)y,$$
$$(c-a)(4a-3z^2)^{\frac{1}{2}} \pm (c-b)(4b-3z^2)^{\frac{1}{2}} = 3(a-b)z. \ldots\ldots 194$$

8333. (Professor Hanumanta Rau, B.A.)—Prove that the equations $x^3+19x-40=0$ and $7x^4-12x^3+4bx^2+12x+7=0$ have two common roots. 195

8341. (J. Brill, M.A.)—If ABCDEF be a hexagon inscribed in a circle, prove that

$\Delta BCD \cdot \Delta EFA - \Delta CDA \cdot \Delta EFB + \Delta DAB \cdot \Delta EFC - \Delta ABC \cdot \Delta EFD = 0.$

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8343. (Chas. E. McVicker, M.A.)—If ρ be the radius of the circle which cuts orthogonally the escribed circles of any triangle, and ρ_1, ρ_2, ρ_3 the radii of the circles cutting similarly the inscribed and two of the escribed circles; prove that

(1) $4\rho^2 = \frac{(b+c)(c+a)(a+b)-abc}{a+b+c} = r^2 + \frac{r_1r_2r_3}{r}$, (2) $\rho^2+\rho_1^2+\rho_2^2+\rho_3^2 = 4R^2$.

..... 137

8353. (Professor Booth, M.A.)—Eliminate (1) x, y, z from the equations $ax^2+2hxy+by^2=0$, $by^2+2fyz+cz^2=0$, $cz^2+2gzx+ax^2=0$, and give (2) a geometrical interpretation of the result. 162

8370. (Professor Mukhopâdhyây, M.A., F.R.S.E.)—A piece of cardboard is 8 inches by 12 inches. Find how many gun-wads $\frac{3}{4}$ inch in diameter can be punched out of it. 195

8377. (Asparagus.)—The normal at any point p of a given ellipse meets the axes in G, g; Gg is bisected in O, and from O are drawn three other normals (always real) OP, OQ, OR; prove that P, Q, R will be the corners of a maximum triangle inscribed in the ellipse, and the circles of curvature at P, Q, R will concur in a point p' of the ellipse such that pp' is a diameter. 51

8413. (F. Morley, B.A.)—If OPSQ, $OP_1S_1Q_1$ are focal chords of an ellipse, show that the four points in which SP_1, SQ_1 meet S_1P, S_1Q, lie on the confocals through O, two on each..... 195

8419. (Asparagus.)—At any point P of a given ellipse is drawn the radius of curvature PI, and a circle described on PI as diameter; this circle meets the ellipse again in Q, Q': prove that the tangents to the ellipse at Q, Q' meet in a point O on the circle; that the locus of O is the inverse of the ellipse with respect to the centre; and the first negative pedal of the locus of O is an ellipse similar to the given ellipse. 138

8429. (H. J. Read, B.A.)—Transform, by means of the geometry of the circle, the integrals

$$\int_0^{2\pi} \frac{d\phi}{(1+a^2-2a\cos\phi)^{\frac{1}{2}}}, \qquad \int_0^{2\pi} \frac{\cos\phi\, d\phi}{(1+a^2-2a\cos\phi)^{\frac{1}{2}}}.$$

(See Cheyne's *Planetary Theory*, pp. 50 *sqq.*) 93

8447. (Emile Vigarié.)—Etant donnés deux cercles de centre O et O', trouvez le lieu des points P tels que sa polaire par rapport au premier cercle soit tangente au second. 59

8457. (W. J. C. Sharp, M.A.)—If E be the spherical excess of a triangle whose sides are a, b, c; show (1) that

$$\sin^2\tfrac{1}{4}E = \frac{\sin\frac{1}{2}s\sin\frac{1}{2}(s-a)\sin\frac{1}{2}(s-b)\sin\frac{1}{2}(s-c)}{\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c},$$

where $2s = a+b+c$; and hence (2) deduce the ordinary formula for the area of a plane triangle..... 196

8459. (Piers C. Ward, M.A.) — Prove, without expansion, the equality of the determinants

$$\begin{vmatrix} a+b+c & 3a & 3a \\ 3b & a+b+c & 3c \\ 3b & 3b & a+b+c \end{vmatrix} = \begin{vmatrix} a+b+c & 3c & 3c \\ 3b & a+b+c & 3c \\ 3b & 3b & a+b+c \end{vmatrix}. \quad 197$$

8470. (Professor Wolstenholme, M.A., Sc.D) — The circle of curvature is drawn to an ellipse at a point P whose focal distances are as $a^{\frac{1}{2}}+(a-b)^{\frac{1}{2}} : a^{\frac{1}{2}}-(a-b)^{\frac{1}{2}}$; PR is the common chord of the ellipse and circle, and meets QQ′, the common tangent, in the point O; prove that $OP = a+b$, $OQ = OQ' = a-b$, $2a$, $2b$ being the lengths of the axes of the ellipse..... 140

8472. (Professor Sircom, M.A.) — Show that the value of x from the equation $10^x = x+1$ is $x = \cdot 13712916$..... 197

8536. (Professor Hanumanta Rau, B.A.)—A, B are two points on the diameter of a circle equidistant from the centre. AP, BQ are any two parallels meeting the circumference. Find the locus of the points of intersection of the tangents at P and Q. 64

8546. (R. Knowles, B.A.)—Tangents from a point T (hk) meet the ellipse $a^2y^2+b^2x^2 = a^2b^2$ in P and Q, a third tangent at R meets these in M, N respectively; show that, if θ be the eccentric angle of the point R, the area of the triangle TMN is

$$(a^2k^2+b^2h^2-a^2b^2)^{\frac{1}{2}}\,(ak\sin\theta+bh\cos\theta-ab)\,/\,(ak\sin\theta+bh\cos\theta+ab).$$

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8567. (Rev. T. P. Kirkman, M.A., F.R.S.) — If, in any 4-edron $abcd$, ca be the 2-edral angle about ac, prove that

$$\left[\cos ca+\cos(cb-cd)\right]\left[(bd)^2-(cb-cd)^2\right] = \left[\cos ca+\cos(ad-ab)\right]\left[bd^2-(ad-ab)^2\right]..... 162$$

8569. (J. O'Byrne Croke, M.A.) — Prove that the square of the tangent of half the vertical angle of an isosceles triangle in terms of the lengths of the segments s_1, s_2; σ_1, σ_2, of any two straight lines drawn through the middle point of the base and terminated by the sides, is equal to $\left\{\left(\frac{1}{s_1}-\frac{1}{s_2}\right)^2-\left(\frac{1}{\sigma_1}-\frac{1}{\sigma_2}\right)^2\right\}\Big/\left\{\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}\right)^2-\left(\frac{1}{s_1}+\frac{1}{s_2}\right)^2\right\}$... 32

8582. (D. Edwardes.)—If SY, SZ, S′Y′, S′Z′ are the focal perpendiculars on tangents to an ellipse from a point O; prove that (1) YZ, Y′Z′, and a perpendicular from O on major axis are concurrent; and (2) if YZ meet OS′ in L, and Y′Z′ meet OS in M, the points L, M, S, S′ are concyclic..... 139

8586. (Professor Sylvester, F.R.S.) — Assuming that the pressure of fluids is transmitted equally in all directions, prove the following theorem in Statics:—Any system of forces respectively perpendicular and proportional to the faces of a polyhedron is in equilibrium. 163

8599. (Professor Sircom, M.A.) — If $\frac{d}{dx} \equiv \mathrm{D}$, prove that

$$\left\{x^{-pr}\,(x^{p+1}\mathrm{D})^r\,x\right\}^p \phi(x) = x^p\,\mathrm{D}^{pr}\left\{x^{pr}\,\phi(x)\right\}. ..... 97$$

8610. (S. Tebay, B.A.)—If $\sum_1^\infty \frac{1}{i^{2n}} = c_{2n}\, u^{2n}$, and a_{2n} be the coefficient of θ^{2n} in the expansion of $\theta \cot \theta$, show that $a_{2n} + 2c_{2n} = 0$. 97

8612. (Maurice d'Ocagne.) — Soient d et d' les cordes réelles communes à une conique C et à un point O considéré comme cercle de rayon nul. Du point M, pris sur l'une ou l'autre des droites d et d', menons à la conique C les tangentes MT et MT', dont les points de contact sont T et T', et prenons sur ces tangentes les points N et N' tels que les angles TON et T'ON' soient droits. La droite NN' coupe en I la polaire du point O par rapport aux tangentes MT et MT'. Démontrer que la droite IO passe par le point de rencontre des droites d et d'..... 139

8620. (F. Morley, B.A.)—If a transparent visible cube be divided into n^3 equal cubes by transparent visible planes, show that in the figure so formed (1) the number of visible cubes is $\{\frac{1}{2}n(n+1)\}^2$, or is equal to the number of visible rectangles in any face of the figure; (2) the number of visible parallelepipeds (including cubes) is $\{\frac{1}{2}n(n+1)\}^3$. [See Vol. XLIV., p. 109, Quest. 8340.] 64

8629. (Asparagus.)—Three normals OP, OQ, OR are drawn from a point O to a parabola whose focus is S; and OP, OQ make complementary angles with the axis; prove that SO is parallel to the tangent at R. 62

8642. (Professor Mathews, M.A.) — Give an elementary geometrical method for constructing the spheres which pass through two given points, and touch two given planes. 31

8652. (R. Curtis, M.A.) — If $x_1y_1z_1$, $x_2y_2z_2$, $x_3y_3z_3$, $x_4y_4z_4$ are the coordinates of the four vertices of a tetrahedron (axes of coordinates being oblique), and ϕ = the solid angle contained by the three coordinate planes, show that

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \times \frac{\sin\phi}{6} = \text{volume of tetrahedron.} \ldots\ldots 117$$

8653. (S. Tebay, B.A.)—Trains are dispatched simultaneously at equal intervals from two stations on the same line, in opposite directions, and with equal speeds. If n trains be dispatched from each station after the first two meet, find the number of conjunctions that have taken place when all are clear. 61

8654. (F. Morley, M.A.)—There are n towns joined two and two by railroads. Show that it is possible to pass by rail over the entire system without using any twice if n be odd; and impossible if n be even. 84

8665. (Rev. T. R. Terry, M.A.) — If the conjugate quaternion Kq is defined by the equation $q + \mathrm{K}q = 2\mathrm{S}q$, prove that

(1) $\mathrm{K}(qr) = \mathrm{K}r \,.\, \mathrm{K}q$, and (2) $rq + qr + r^{-1}(rq + qr)\,\mathrm{K}r = 4q\mathrm{S}r$. ... 58

8666. (R. Tucker, M.A.)—Solve the equation

$$b^2z^3 - (a^2 - b^2)\,z^2 + (a^2 - c^2)\,z - c^2 = 0,$$

where a, b, c are sides of a triangle..... 197

8684. (Professor Mahendra Nath Ray, M.A., LL.B.) — An indefinite straight line envelopes a fixed parabola whose equation, when referred to the axes of a given series of confocal ellipses, is $(x)^{\frac{1}{2}} + (y)^{\frac{1}{2}} =$ constant. Show that the locus of the poles of the straight line with respect to the confocal ellipses always passes through a fixed point. 140

8691. (Professor Hanumanta Rau, B.A.)—Prove that

$$\int_0^{\frac{1}{2}\pi} \cos^6\theta \cos^2 5\theta \, d\theta = \frac{5\pi}{64}. \quad \text{..... 82}$$

8693. (R. Tucker, M.A.)—If P, Q, R are three points on a parabola, such that $\Sigma y^{-1} = 0$, then the centres of curvature at these points are collinear. If they are three points which have a co-normal point (O), then the radical centre (C) of the circles of curvature will lie on the directrix, if the locus of O is a certain straight line parallel to the directrix. If the locus of O is a certain hyperbola, the polar of C with regard to the parabola passes through O; and if the locus is a certain semicubical parabola, C will lie on the parabola..... 112

8698. (Asparagus.) — The circle of curvature to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at a point P is drawn, and QQ′ is the common tangen to the ellipse and circle; prove that, if e be the eccentricity of the ellipse, and θ the excentric angle of the point P, then

$$QQ' = \frac{8ae^2 \sin^3\theta \cos^3\theta (1-e^2\cos^2\theta)^{\frac{1}{2}}}{1-2e^2\cos^4\theta(1+2\sin^2\theta)+e^4\cos^6\theta(1+3\sin^2\theta)}. \quad \text{..... 140}$$

8702. (R. Knowles, B.A.)—Prove that

$$\int x^{n-1}\log(1+x)\,dx = \frac{1}{n}\left(x^n \int \frac{dx}{1+x} - \int \frac{x^n dx}{1+x}\right). \quad \text{..... 50}$$

8703. (S. Tebay, B.A.)—The distances x, y, z of a point O from the angles of a triangle form the base of a tetrahedron whose conterminous edges a, b, c are respectively opposite to x, y, z. Find the position of O when the volume is a maximum. 92

8704. (A. Gordon.)—If a rod is hung horizontally by two parallel strings at its extremities, and displaced slightly, (1) along the direction of the rod, (2) by a small rotation round the vertical through its centre, prove that the ratio of the times of oscillation is constant and $= 3^{\frac{1}{2}} : 1$. 41

8706. (Rev. T. R. Terry, M.A.)—Find the eccentricity of the conic

$$27x^2 - 24xy + 20y^2 - 30x - 88y + 131 = 0,$$

and determine the position of its foci and directrices..... 114

8707. (Professor Mukhopâdhyây, M.A., F.R.S.E.)—Show that the reciprocal polar of the evolute of the evolute of the conic $x^2/a^2 + y^2/b^2 = 1$, with respect to the circle described on the line joining the foci as diameter, is the curve $a^2x^2 + b^2y^2 = (a^2y^4 - b^2x^4)^{\frac{1}{2}}$. 141

8720. (Professor Edmund Bordage.) — Three points A, B, C and a straight line XY are given in a same plane. Determine the points of intersection of the line XY with the circle passing through the three points A, B, C, without drawing the circle. 59

8730. (J. Brill, M.A.)—If the triangles ABC and DEF have the same centroid, prove that

$$\frac{AD\,.\,AE\,.\,AF\,.\,BC}{\sin(BDC+BEC+BFC-BAC)} = \frac{BD\,.\,BE\,.\,BF\,.\,CA}{\sin(CDA+CEA+CFA-CBA)} = \frac{CD\,.\,CE\,.\,CF\,.\,AB}{\sin(ADB+AEB+AFB-ACB)},$$

where BDC denotes the angle through which DB must be turned in the positive direction in order to coincide with DC, and so on. 94

8734. (Maurice d'Ocagne.) — On donne deux points F et P et une droite Δ parallèle à FP. Si du point P on mène des tangentes t et t' à une parabole variable π, de foyer F, tangente à Δ, 1° la polaire du point F par rapport aux droites t et t' est une droite fixe δ; 2° la polaire du point P par rapport à la parabole π passe par un point fixe situé sur δ. 142

8736. (H. G. Dawson, M.A.)—P, P′, Q, Q′, R, R′ are six points on a conic; PP′, QQ′ meet in C; QQ′, RR′ in A; RR′, PP′ in B. Show that, if a quartic curve be described through P, P′, Q, Q′, R, R′, A, B, C, and if the tangents to it at ABC make angles $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$ with the sides AC, AB, BA, BC, CB, CA, then $\sin\alpha \sin\beta \sin\gamma = \sin\alpha' \sin\beta' \sin\gamma'$, and hence that a conic can be drawn through A, B, C touching the quartic in A, B, C. 81

8738. (Rev. T. R. Terry, M.A.) — AOB is a straight line, AO = a, OB = b. With O as focus, a family of conics is described passing through the points A, B, each conic touching the preceding member of the family. If e is the eccentricity, show that the values of the expression $[(a+b)^2e^2-(a-b)^2]^{\frac{1}{2}}$ for the different conics form an arithmetic progression, whose common difference is $2(a+b)$. 100

8744. (D. Edwardes.) — A sphere of radius a is surrounded by an infinite mass of incompressible fluid. If the surface of the sphere be suddenly moved with normal velocity $eyz+fzx+gxy$ at any point, prove that the velocity potential of the resulting initial motion is $-a^6(eyz+fzx+gxy)/(3r^5)$, where r is the distance of any point from the centre. 92

8750. (Professor Mathews, M.A.)—Denoting by $[x]$ the greatest integer contained in the numerical quantity x, prove that, if a, b are positive integers prime to each other,

$$\left\{\frac{b}{a}\right\}+\left\{\frac{2b}{a}\right\}+\left\{\frac{3b}{a}\right\}+\ldots\left\{\frac{(a-1)\,b}{a}\right\}$$
$$=\left\{\frac{a}{b}\right\}+\left\{\frac{2a}{b}\right\}+\left\{\frac{3a}{b}\right\}+\ldots\left\{\frac{(b-1)\,a}{b}\right\}=\tfrac{1}{2}(a-1)(b-1).... 105$$

8753. (Professor Hudson, M.A.)—ABC is a right-angled triangle; a point D is taken in the hypothenuse BC, and circles are described about ABD, ACD. If E, F be the mid-points of the arcs AB, AC remote from D, prove that EAF is a straight line..... 27

8755. (Professor Neuberg.) — On prolonge les hauteurs du triangle ABC au-delà des sommets des quantités AA′ = BC, BB′ = CA, CC′ = AB; démontrer : (1) les triangles ABC, A′B′C′ ont le même centre de gravité;

(2) si α, α' sont les angles de Brocard de ces triangles, on a
$A'B'C' = 2ABC(2 + \cot\alpha)$, $(A'B')^2 + (B'C')^2 + (A'C')^2 = 4ABC(3 + 2\cot\alpha)$,

$$\cot\alpha' = \frac{2\cot\alpha + 3}{2\cot\alpha + 4};$$

(3) les points A, B, C sont les centres des carrés construits, intérieurement, sur les côtés du triangle A'B'C'; (4) les milieux des côtés de A'B'C' sont les centres des carrés construits, extérieurement, sur les côtés de ABC. 78

8756. (Professor Wolstenholme, M.A., Sc.D.)—PSQ is a focal chord of a conic, and from S', the other focus, perpendiculars are drawn upon the normals at P, Q; prove that the straight line joining the feet of these perpendiculars will be parallel to PQ, and will pass through the centre of the conic. 40

8758. (Professor Ignacio Beyens.)—Résoudre le système d'équations

$$a^x(a^y + a^z + a^{x+y+z}) - b = 0, \quad a^y(a^x + a^z + a^{x+y+z}) - c = 0,$$
$$a^z(a^x + a^y + a^{x+y+z}) - d = 0. \ldots\ldots 198$$

8772. (Rev. T. R. Terry, M.A.)—If C is the centre of the conic

$$f(x, y) \equiv ax^2 + 2c'xy + by^2 + 2b'x + 2a'y + c = 0,$$

and P is any point (h, k), and if CP, produced if necessary, meets the conic in Q; show (1) that

$$(ab - c'^2)\,CQ^2 f(h, k) = (abc + 2a'b'c' - aa'^2 - bb'^2 - cc'^2)(CQ^2 - CP^2);$$

and (2) illustrate this result from *à priori* considerations. 65

8773. (Professor Mukhopâdhyây, M.A., F.R.S.E.)—Show that the reciprocal polar of the evolute of the evolute of the parabola $y^2 = 4a(x + 2a)$, with respect to a circle of radius a, is the curve $y^2(3x^2 + 2y^2) = ax^3$... 141

8776. (Asparagus.)—From a point O are drawn four normals OP, OQ, OP', OQ' to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and OP, OQ make complementary angles with the axes; prove that (1) the locus of O is the curve

$$(x^2 - y^2)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = (a^2 - b^2)\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2;$$

(2) the envelope of PQ is the hyperbola $\left(\frac{x^2}{a^4} - \frac{y^2}{b^4}\right)(a^2 - b^2) = 1$;

and (3) P'Q' will be normal to the rectangular hyperbola $4(x^2 - y^2) = a^2 - b^2$. 142

8785. (Maurice d'Ocagne.)—Si les points A et B se déplacent sur la directrice D d'une conique C, dont F est le foyer correspondant, de telle façon que l'angle AFB soit constamment droit, le quadrilatère formé par les tangentes menées des points A et B à la conique C, reste inscrit dans une conique ayant le point F pour foyer et la droite D pour directrice correspondante. 143

8791. (Professor Genese, M.A.)—If a conic circumscribes a parallelogram OABC, and a variable straight line through O meets the opposite sides in P, Q, and the curve in R; prove that

PR : RQ varies as $OP^2 : OQ^2$, or as AP : CQ. 143

8794. (Professor Ignacio Beyens.)—Un bon tirailleur *aux pigeons* a la probabilité de tuer le $66\frac{2}{3}$ *per cent* des pigeons; il se propose de tirer 3^{10}

dizaines d'oiseaux; lesquels sont les dizaines où il tuera un pigeon, lesquels sont le nombre des dizaines où il tuera deux pigeons, trois, etc ? 62

8805. (R. Curtis, M.A.) — Find (1) the locus of a point in a material lamina, such that if the lamina were moving without rotation in its own plane, and that point were suddenly fixed in space, the ensuing rotation of the lamina would be given; (2) where the point should be if the rotation produced is a maximum; and (3) where the point should be made that the loss of kinetic energy would be given. 67

8809. (J. J. Walker, M.A., F.R.S.)—Show that the time of sliding down any chord of a central conic, with axis vertical, to the lowest vertex, is equal to the time of falling down a height equal to the parallel focal chord. 144

8811. (A. Russell, B.A.) — If the temperature of an infinite cube of isotropic matter be initially zero, and the faces be subsequently kept at temperatures $A_1 \sin \frac{\pi y}{2l} \sinh \frac{\pi z}{2l} f_1(t)$, $A_2 \sin \frac{\pi z}{2l} \sinh \frac{\pi x}{2l} f_2(t)$, &c.,

the axes being three edges of the cube; find the temperature at any subsequent period. 34

8812. (R. W. D. Christie.) — If a_n be the coefficient of x^n in the development of $1/(1+2x-x^2)$, prove that

$$(2m+1)\pi + \tfrac{1}{4}\pi = 2\tan^{-1}\frac{a_{n+1}}{a_n} + (-1)^n \tan^{-1}\frac{1}{a_n^2 + 2a_n a_{n+1} - a_{n+1}^2},$$

where m is a positive or negative integer or zero. 199

8817. (W. J. Greenstreet, B.A.)—In the "ambiguous case," given a, b, and A; ν, ν' being radii of in-circles and ρ, ρ' radii of ex-circles to a, prove that $\nu\nu'(b+a) = \rho\rho'(b-a)$. 81

8821. (Rev. T. C. Simmons, M.A.)—If ABC be a *positively lettered* triangle (*i.e.*, drawn so that, in moving round the triangle from A to B, B to C, C to A, we revolve *positively*), whose *positive* and *negative* Brocard-points are ω, ω', and Lemoine-point K; and if AK, $A\omega$, $A\omega'$ meet BC in X, x, x'; and Y, y, y', Z, z, z' be similarly taken on CA, AB; prove that $z'Y$, Zy are both parallel to BC, and similarly for the corresponding lines. Hence, having given the Lemoine-point of a triangle, show how the Brocard-points may be easily determined. 61

8822. (Professor Sylvester, F.R.S.) — Let U be any cubic surface, Θ the covariant to U of the 9th order in the variables and 11th degree in the coefficients which contains the 27 ridges of U. Prove that $x^6\Theta - 2x^3 U\frac{d}{da}\Theta + U^2\frac{d^2}{da^2}\Theta$ is a scroll which contains the 27 ridges of $U + \lambda x^3$ for every value of λ. 46

8825. (Professor Genese, M.A.) — If S, S′ be isogonal points of a triangle ABC (*i.e.*, foci of an in-conic), the conic ABCSS′ touches each of the sixteen conics which can be drawn through S, S′ to touch the sides of the four triangles determined by the bisectors of the angles A, B, C, taking one at a time through each vertex. 101

8828. (Professor Nash, M.A.)—Prove that the envelope of the polar of any point with respect to a series of Tucker circles is a parabola whose axis is perpendicular to the line of centres of the circles. 144

8831. (Professor Hudson, M.A.) — Prove that the perpendicular from the orthocentre on the bisector of an angle of a triangle meets it on that diameter of the nine-points circle that passes through the mid-point of the opposite side. 36

8836. (Professor Wolstenholme, M.A., Sc.D.) — A chord of the parabola $y^2 = 4ax$ is drawn through the focus S and through a point P $[2am, a(1+m^2)]$ on the parabola $y^2 = 4a(x-a)$; prove that, (1) of all chords of the former parabola which can be drawn through P, the chord aforesaid is always either a maximum or a minimum; a maximum if $SP > 6a$, and a minimum if $SP < 6a$; (2) when this chord is a maximum there are also two real minimum chords, each of the same length $4am$, and inclined to the axis at complementary angles; the length of the maximum chord being $a(4+m^2)$; and (3) find the envelope of the straight lines on which the minimum chords lie. 145

8841. (L. J. Rogers, B.A.) — Show that the complete primitive of the reciprocant equation

$$9\left(\frac{dy}{dx}\right)^2 \cdot \frac{d^4y}{dx^4} - 45\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}\cdot\frac{d^3y}{dx^3} + 40\left(\frac{d^2y}{dx^2}\right)^3 = 0$$

is $(x^2+2px+q)(y^2+2ry+s) = (p^2-q)(r^2-s)$. 34

8848. (Asparagus.) — A circle is drawn through a fixed point O, touching a fixed circle internally or externally, and meeting in P, P′ the two tangents drawn from O to the fixed circle; prove that the envelope of PP′ is a circle touching the two tangents drawn from O to the fixed circle and having its centre at the middle point of their chord of contact. 35

8849. (R. W. D. Christie.) — Prove that Bernoulli's numbers are of the form $\frac{6N \pm 1}{6n}$. 199

8864. (Professor Sylvester, F.R.S.)—Extend Bedetti's theorem concerning the intersection of a surface with a tangent plane, by showing that the intersection of a hyper-surface with its linear and quadratic polars in respect to any given point upon it, is a curve of double curvature having a sextuple point at the given point. 69

8872. (Professor Ignacio Beyens.)—Si dans un triangle on voit, du centre du cercle inscrit, sous un angle droit, la distance entre l'orthocentre et le circumcentre, démontrez que $a^2+b^2+c^2 = 4(2R-r)(R+r)$. 43

8877. (Professor Nilkantha Sarkar, M.A.)—Find (1) the interest with which any capital whatever would be increased at the end of a year, it the interest due at every instant (*i.e.*, second) of the year were itself to become capital and bear interest; and (2) what, if the increase of capital were regarded as constant. 200

8885. (Rev. T. W. Openshaw, M.A.) — Through the cosinusoid $AeCf B$ having the abscissa $OB = \frac{1}{2}\pi$, and the extreme ordinate

$$OA = \cos 0 = 1, \quad OB = \cos \tfrac{1}{2}\pi = 0,$$

draw a straight line $aefb$ so that the area $Aae + egfC + fbB$ shall be a minimum. 109

8889. (Rev. T. R. Terry, M.A.) — A set of rectangular axes is brought from one given position to another by a rotation round the common origin. Show how to find the axis and angle of rotation in the case in which the second position relative to the first is given by the equations $4x' = x + y(1-2\sqrt{2}) + z(2+\sqrt{2})$,

$4y' = x(1+2\sqrt{2}) + y + z(\sqrt{2}-2)$, $4z' = x(\sqrt{2}-2) + y(\sqrt{2}+2) + 2z$. 200

8890. (Major H. W. L. Hime, R.A.)—If ABC be any triangle, I the in-centre, P the orthocentre and Q the circumcentre, and if AI meets PQ in J, prove (1) that $PJ/JQ = 2\cos A$; and hence (2) that if $A = 60°$ J is the mid-centre (*i.e.*, centre of the nine-point circle). 118

8894. (J. M. Dyer, M.A.)—If p is a prime >3, show that

$$(p-1)!\ [\tfrac{1}{1} + \tfrac{1}{2} + \ldots\ldots 1/(p-1)]$$

is divisible by p^2. [Or, thus:—$d/dp\,(p-1!) \equiv M(p^2)$.] 115

8896. (F. Morley, B.A.) — Prove that, (1) the inverse of either centre of similitude of two orthogonal circles with respect to either circle is the inverse of the other centre with respect to the other circle; and (2) the two inverses and the two intersections of the circles are the angular points of a square. 98

8899. (Professor Hanumanta Rau, B.A.)—If N is any number in the scale whose radix is 13, 14, or 15, N ends in the same digit as N^{13}.... 52

8911. (Professor Byomakesa Chakravarti, M.A.) — The tangents at A, A′, extremities of a diameter of an ellipse, meet the tangent at P in Y, Y′; CB is the semi-diameter conjugate to CA; prove that the areas of the triangles YCY′, ABA′, APA′ are in geometrical progression... 50

8913. (Professor Hudson, M.A.)—If $\sin(x+h) = \sin x + h\cos(x+\theta h)$ prove that the coefficient of h^3 in the expansion of θ in powers of h is

$$\tfrac{1}{6\,7\,8\,0}\left\{57\cot x + 55\cot^3 x\right\}. \ldots\ldots 45$$

8917. (Maurice D'Ocagne.) — On prend dans un cercle un diamètre AB et une corde CD parallèle à AB. La droite AD coupe en M la tangente menée au cercle par le point B. Démontrer que, si P est le pied de la perpendiculaire abaissée du point M sur la droite AC, (1) on a $AP = 2AC$; et (2) lorsque la corde CD varie dans le cercle en restant parallèle au diamètre AB le lieu du point P est le cercle tangent en A au cercle donné et de diamètre double de celui-ci. 42

8918. (Professor Mahendra Nath Ray, M.A., LL.B.)—If $A^{\frac{1}{2}} = (A - 2ax - ay + a^2)^{\frac{1}{2}} + (A - ax - 2ay + a^2)^{\frac{1}{2}}$, where $A = x^2 + xy + y^2$; show (1) that $A = a(x+y)$, and (2) give a geometrical interpretation. 56

8921. (H. L. Orchard, B.Sc., M.A.)—Solve the equation

$$2x^6 + 6x^5 + 5x^4 - x^3 - 53100 = 0. \ldots\ldots 80$$

8922. (Major Hime, R.A.)—If N is the centre of the nine-point circle of the triangle ABC, and if AN produced meets BC in A′, prove that

$$\frac{AN}{NA'} = 1 + \frac{2\cos A}{\cos(B-C)}, \quad \frac{BA'}{A'C} = \frac{\sin 2A + \sin 2B}{\sin 2A + \sin 2C}. \ldots\ldots 68$$

8925. (F. Morley, B.A.)—Prove that, in any acute-angled scalene triangle, $\tan A \tan B \tan C > 3\sqrt{3}$. 68

8927. (S. Tebay, B.A.)—If Δ_1, Δ_2, Δ_3, Δ_4 be the areas of the face, of a tetrahedron, R the radius of the circumscribing sphere, and R_1, R_2s R_3, R_4 the radii of spheres passing through the centre of R and the angles of Δ_1, Δ_2, Δ_3, Δ_4; the volume

$$V = \tfrac{1}{6}R^2\left(\frac{\Delta_1}{R_1}+\frac{\Delta_2}{R_2}+\frac{\Delta_3}{R_3}+\frac{\Delta_4}{R_4}\right).$$ 166

8932. (R. Curtis, M.A.)—Supposing the discriminant of $S_1 + kS_2$ to be written thus, $k^3\Delta_1 + k^2\Theta_{12} + k\Theta_{21} + \Delta_2$, where

$$S_1 \equiv (a_1b_1c_1f_1g_1h_1)(xyz)^2, \quad S_2 \equiv (a_2b_2c_2f_2g_2h_2)(xyz)^2;$$

T the invariant for S_1, S_2, S_3 $(a_1b_2c_3)^2$ + &c. (see Salmon's *Conics*, Art. 689 6th Ed.); and $T' = (A_1B_2C_3)^2$ + &c., the corresponding invariant for the tangential equation; prove that the condition that S_1, S_2, and S_3 should have a common triangle self-conjugate is

$$TT' = \begin{vmatrix} 3\Delta_1 & \Theta_{12} & \Theta_{13} \\ \Theta_{21} & 3\Delta_2 & \Theta_{23} \\ \Theta_{31} & \Theta_{32} & 3\Delta_3 \end{vmatrix}^2.$$ 66

8936. (A. Russell, B.A.)—Prove that the respective equations to the ellipsoid and the wave surface may be written in the forms

$$\begin{vmatrix} 1-\frac{a^2}{x^2}, & 1, & 1 \\ 1, & 1-\frac{b^2}{y^2}, & 1 \\ 1, & 1, & 1-\frac{c^2}{z^2} \end{vmatrix} = 0, \quad \begin{vmatrix} \frac{a^2-y^2-z^2}{x^2}, & 1, & 1 \\ 1, & \frac{b^2-z^2-x^2}{y^2}, & 1 \\ 1, & 1, & \frac{c^2-x^2-y^2}{z^2} \end{vmatrix} = 0.$$

..... 82

8944. (W. J. C. Sharp, M.A.)—Form the equation to the two tangent planes, which can in general be drawn to a given quadric, so as to contain a given straight line. 28

8965. (W. J. C. Sharp, M.A.)—Show that, if

$$x^n - P_1x^{n-1} + P_2x^{n-2} - P_3x^{n-3} + \&c. = 0$$

be the equation whose roots are the squares of those of

$$x^n - p_1x^{n-1} + p_2x^{n-2} - p_3x^{n-3} + \&c. = 0,$$

$$P_r = \frac{1}{2(n-r)!}\left\{\frac{p_{n-1}}{p_n}\left(\frac{d}{dp_1}+p_1\frac{d}{dp_2}+p_2\frac{d}{dp_3}+\&c.\right)\right.$$
$$\left.-\left(n\frac{d}{dp_2}+(n-1)p_1\frac{d}{dp_3}+(n-2)p_2\frac{d}{dp_4}+\&c.\right)\right\}^{n-r}(p_n^2).$$

..... 120

8976. (W. J. C. Sharp, M.A.)—Show that, if (1.2), (2.3), &c. denote the edges of a simplicissimum in space of n dimensions (see Quest.

8242), the squared radius of the circumscribed hyper-sphere is

$$-\tfrac{1}{2}\begin{vmatrix} 0, & (1.2)^2, & (1.3)^2\ldots \\ (1.2)^2, & 0, & (2.3)^2\ldots \\ \ldots & \ldots & \ldots \\ (1.n+1)^2, & (2.n+1)^2, & (3.n+1)^2\ldots \end{vmatrix} \div \begin{vmatrix} 0, & 1, & 1, & 1\ldots \\ 1, & 0, & (1.2)^2, & (1.3)^2\ldots \\ 1, & (1.2)^2, & 0, & (2.3)^2\ldots \\ & \ldots & \ldots & \ldots \\ 1, & (1.n+1)^2, & (2.n+1)^2, & (3.n+1)^2\ldots \end{vmatrix}$$

which reduces to $\frac{n}{2(n+1)}$ when each of the edges is unity.

Also, show that, if d be the distance of the mean centre of equal masses at the vertices of the simplicissimum from the centre of the hyper-sphere, the same quantity equals $d^2 + \frac{1}{(n+1)^2}\{(1.2)^2 + (2.3)^2 + (3.1)^2 + \&c.\}$.

..... 165

8983. (Professor Mathews, M.A.)—ABCD is a convex quadrilateral of which the sides touch one circle, while the vertices lie on another. Tangents are drawn to the circum-circle at A, B, C, D so as to form another convex quadrilateral; prove that the area of the latter

$$= 2r^2 \frac{(s\sigma - 2abcd)(abcd)^{\frac{1}{2}}\sigma}{s(\sigma - bcd)(\sigma - cda)(\sigma - dab)(\sigma - abc)},$$

where r is the radius of the circle ABCD; a, b, c, d are the lengths of AB, BC, CD, DA, respectively;

$$2s = a + b + c + d,\quad 2\sigma = bcd + cda + dab + abc. \ldots\ldots 146$$

8984. (Professor Hudson, M.A.) — Prove that the locus of the feet of the perpendiculars from the origin on the chords of $xy = a^2$ (rectangular axes) that subtend half a right angle at the origin is

$$r^4 - 2a^2r^2 \sin 2\theta = a^4 \ldots\ldots 42$$

8988. (Professor Ignacio Beyens.) — Si a, b, c sont les côtés d'un triangle quelconque $(a < b < c)$, α_1, α_2 les bissectrices de l'angle A et son supplémentaire et l'on construit un triangle avec les côtés α_1, α_2 et ayant A pour angle compris; et de la même manière l'on fait avec les bissectrices β_1, β_2 des angles B, γ_1, γ_2 des angles C, et l'on appelle S_1, S_2, S_3 les surfaces de ces triangles, on aura $S_1^{-1} + S_3^{-1} = S_2^{-1}$. 68

8997. (Professor Hanumanta Rau, B.A.)—Solve the equation

$$x^5 + 5ax^4 + 20x^3 + 20ax^2 + 20x + 4a = 0 \ldots\ldots 201$$

9004. (W. J. C. Sharp, M.A.) — A hyper-surface, in space of n dimensions, is intersected by the linear polar of any point upon it in a hyper-surface (in space of $n-1$ dimensions) having a node at the given point. If the hyper-surface be a quadric (in space of four dimensions) the intersection is a quadric cone whose vertex is at the given point. 69

9008. (Samuel Roberts, M.A.) — Given three circles C_1, C_2, C_3, determine a circle cutting C_1 orthogonally, bisecting C_2, and bisected by C_3; and show that in general there are two such circles which may coincide or become imaginary. 29

9015. (Rev. T. C. Simmons, M.A.)—The lengths of three lines are known not to exceed a, b, c, respectively; find the chance of their forming a possible triangle. 74

9021. (R. W. D. Christie.)—Prove that
$$\frac{\tan 52\frac{1}{2}^\circ + \tan 7\frac{1}{2}^\circ}{\tan 82\frac{1}{2}^\circ + \tan 37\frac{1}{2}^\circ} = \frac{\tan 52\frac{1}{2}^\circ - \tan 7\frac{1}{2}^\circ}{\tan 82\frac{1}{2}^\circ - \tan 37\frac{1}{2}^\circ} = \frac{\tan 22\frac{1}{2}^\circ}{\tan 67\frac{1}{2}^\circ} = (\sqrt{2}-1)^2.$$
..... 35

9023. (N'Importe.)—Les 3 côtés d'un triangle forment une progression arithmétique dont on demande la raison ; on connaît le rapport m de la surface de ce triangle à celle du rectangle construit sur les 2 plus petits côtés, calculer les côtés de ce triangle. Discuter. Dans le cas particulier où $m = \frac{1}{2}$, calculer le rayon du cercle inscrit dans ce triangle. 65

9024. (Professor Sylvester, F.R.S.) — For greater distinctness, the name of *Hyper-cartesian* (not to be confounded with a *hyper-cartesic*) being given to that particular form of the bicircular quartic in which four concyclic foci become collinear ; prove that, if four points are given in a plane, the locus of the curve *in space* whose distances from any three of them are subject to a given homogeneous linear relation is a curve of the 4th order. (This spaced curve may be termed a Hyper-cartesic.) ... 75

9036. (Professor Harkoma, M.A.) — A straight rod is divided at random into three parts ; prove that the chance that twice the sum of the squares on the parts shall be less than the square on the whole rod is $\pi/(3\sqrt{3})$. 47

9039. (Professor Byomakesa Chakravarti, M.A.)—If
$$ab\,(ab^{-1}x + a^{-1}by) = (x+y)\,(z^2+xy), \text{ prove that (1)}$$
$$x\,(a+y-z)\,(a+y+z) + y\,(b+x-z)\,(b+x+z) = 2xy\,(a+b+x+y)\ ;$$
and (2)
$$\frac{b+z-x}{b+z+x} \cdot \frac{a+z-y}{a+z+y} = \frac{a+b-x-y}{a+b+x+y}.$$ 123

9040. (Professor Chase, LL.D.) — A point O is taken within a triangle ABC ; P, Q, R are the centres of the circles circumscribing the triangles BOC, COA, AOB, respectively ; determine the position of O that the area PQR may be a minimum. 56

9042. (H. L. Orchard, B.Sc., M.A.)—Prove that $1^3+2^3+3^3+\ldots+x^3$ is a factor of the expression $3x^8+12x^7+14x^6-7x^4+2x^2$. 34

9043. (Rev. T. C. Simmons, M.A.)—In a bag is a particular coin whose value is M, also a number of other coins whose aggregate value is m. A person being allowed to draw one at a time till he draws the coin M, when the drawing is to cease, show that the value of his expectation is $M+\frac{1}{2}m$. 28

9054. (J. Brill, M.A.)—A particle of mass m moves in a conservative field of force subject to the action of a retarding force varying as the velocity ($k \times$ velocity) ; prove that, if D denote the quantity of energy dissipated during a time t, $\frac{dD}{dt} + \frac{2k}{m}(D+V-E) = 0$;
where V denotes the potential energy of the system at time t, and E denotes the total energy of the system when $t = 0$..... 43

9055. (Captain H. Brocard.)—Le cercle de Brocard étant le lieu des centres des symédianes (point de Lemoine) des triangles circonscrits et semblables au triangle donné ABC, démontrer que l'angle ϕ des côtés du triangle T_ϕ avec ceux du triangle ABC n'est autre que l'angle que fait avec OK la ligne qui joint O au centre K'_ϕ des symédianes du triangle T_ϕ. 27

9057. (Emile Vigarié.)—Par les sommets d'un triangle donné ABC, on mène des tangentes a son cercle circonscrit qui déterminent le triangle A'B'C'. On joint les sommets A', B', C' à un point donné M. Les droites MA', MB', MC' coupent les tangentes menées par A, B, C, en α, β, γ. Enfin par les points α, β, γ on mène les tangentes αC''B'', βA''C'', γB''A'' au cercle circonscrit à ABC qui forment le triangle A''B''C''. Démontrer que les droites AA'', BB'', CC'' concourent au point donné M. 80

9059. (A. R. Johnson, M.A.)—In the cubic whose equation is

$$(x^2+y^2)(lx+my+n) = (l'x+m'y+n')^3,$$

show (1) that the product of the distances from the origin of the feet of the nine normals drawn from any point to the cubic, varies as the cube of the distance of the point from the origin; and find (2) what is the analogous theorem in the case of the quartic $(x^2+y^2)w = uv$, where $u=0$, $v=0$, $w=0$ represent any three conics, the two former having one common focus at the origin. 82

9063. (Prof. Mahendra Nath Ray, M.A., LL.B.)—If a_1, a_2, a_3 ... a_{2n-1} be $2n-1$ positive numbers connected by the relation $a_1a_2a_3 \ldots a_{2n-1} = 1$; show, *by elementary algebra only*, that the minimum value of

$$(1+a_1)(1+a_2)(1+a_3)(1+a_4) \ldots (1+a_{2n-1}) \text{ is } 2^{2n-1}. \ldots\ldots 31$$

9064. (A. E. Thomas, M.A.)—If A, B, C are the angles of an acute-angled triangle, prove that $\tan^4 A + \tan^4 B + \tan^4 C \not< 27$,

$$\tan^4 A \tan^4 B + \tan^4 B \tan^4 C + \tan^4 C \tan^4 A \not< 243. \ldots\ldots 35$$

9065. (R. W. D. Christie.)—If $\Sigma^2 = 1^5 + 2^5 + 3^5 \ldots n^5$, show that the first solution in integers is $\Sigma = 1001$, and find another. 30

9066. (Rev. T. R. Terry, M.A.) — Prove the following property of the Gamma Function :—$\Gamma(\frac{1}{2}m)\,\Gamma\frac{1}{2}(m+1) = 2^{1-m}\,\Gamma(\frac{1}{2})\,\Gamma(m)$. 42

9067. (F. Morley, M.A.)—Three particles are connected by two inelastic threads of any length, and placed on a smooth table in a line perpendicular to opposite edges. The outer two, which are of equal mass, are drawn just over the edges and let fall simultaneously. Show that, supposing the inner particle not to leave the table, after the second chuck, all is still. 46

9068. (R. Knowles, B.A.)—Tangents TP, TQ meet a parabola in P and Q; a tangent at R, the point where the diameter through T meets the curve, meets TP, TQ in M and N respectively; prove that the line joining the mid-points of MQ, NP is parallel to PQ..... 44

9070. (R. Curtis, M.A.)—A particle moving freely round a centre of force O is projected into a particle moving freely round a centre of force O'; show that (1) O' is the projection of O; (2) if both orbits be conics, v and F the velocity and force exerted on the particle in the first orbit, v' and F' in the second, α and β the semi-diameters parallel to v and F in first orbit, α' and β' those parallel to v' and F' in the second orbit, $v/\alpha = v'/\alpha'$, $F/\beta = F'/\beta'$; (3) hence prove Sir W. R. Hamilton's theorem, that the law of force for which a particle will describe an ellipse round any point S' in its plane as centre of force is $F = \dfrac{u \cdot \beta^6}{SP^2 \cdot PQ^3}$, where P is the particle and PQ a chord drawn through S and terminated by the ellipse, and α a semi-diameter parallel to PQ..... 29

9071. (Professor Sylvester, F.R.S.)—If A, B, C, D be the four points mentioned in Quest. 9024, F, G, H, K four other points in which the hyper-cartesic cuts the plane passing through the first named four, a bicircular quartic may be drawn through A, B, C, D, of which F, G, H, K will be concyclic foci. 75

9072. (Ch. Hermite, LL.D., Membre de l'Institut.)—On demande de prouver que l'équation transcendante

$$e^{2x} = \{(x+a+1)^2+1\} / \{(x+a-1)^2+1\},$$

où a désigne une constante réelle quelconque, n'admet qu'une seule et unique racine réelle de même signe que a, et moindre en valeur absolue que le logarithme népérien de $\sqrt{2}+1$. 21

9074. (Professor Nilkantha Sarkar, M.A.) — Show that the reciprocal polar of the evolute of the parabola $y^2 = 4ax$, with respect to a circle of radius a, is the cubic $x^3 = y^2(a-2x)$. 58

9077. (Professor Steggall, M.A.)—Show that there are twelve ways of placing eight queens on a chess-board, so that none are *en prise*; and that, of these, six are mutually derivable by uniform horizontal or vertical shifting, four are absolutely independent, and one is semi-symmetrical. 26

9078. (Professor Ignacio Beyens.)—Si les axes des coordonnées OX, OY, OZ sont tels que $\hat{XOY} = \hat{XOZ} = \hat{YOZ} = 60°$, et si $yz+xy+xz+a^2 = 0$ est l'équation d'une surface rapportée à ces axes, et $x+y+z = 0$ un plan dans le même système des coordonnées, l'intersection du plan et de la surface est un cercle de rayon a..... 40

9080. (Professor Neuberg.)—Soient A_n le centre de gravité des points $A_0, A_1, \ldots A_{n-1}$ chargés des masses $a_0, a_1 \ldots a_{n-1}$; A_{n+1} le centre de gravité des points $A_1, A_2, \ldots A_{n-1}, A_n$ chargés des masses $a_0, a_1 \ldots a_{n-2}, a_{n-1}$; A_{n+2} celui des points $A_2, A_3 \ldots A_{n+1}$ chargés respectivement des mêmes masses $a_0, a_1 \ldots a_{n-1}$; et ainsi de suite. Trouver la limite des points $A_n, A_{n+1} \ldots$ 50

9083. (Professor Kalipada Basu, M.A.)—Solve the equation

$$\cos^{-1}\left(1-\frac{a}{x}\right) = \frac{2a}{(2ax-a^2)^{\frac{1}{2}}}. \quad \ldots\ldots 56$$

9090. (S. Tebay, B.A.)—If a, b, c be conterminous edges of a tetrahedron, x, y, z the opposite edges; A, B, C, X, Y, Z the dihedral angles respectively opposite to these edges; A_1, A_2, A_3, A_4 the areas of the faces, and $M = (1-\cos^2 X-\cos^2 Y-\cos^2 Z-2\cos X\cos Y\cos Z)^{\frac{1}{2}}$;

prove that $V = \frac{1}{3}(2MA_1A_2A_3)^{\frac{1}{2}} = \frac{2}{3}A_4^2/(x\cot A+y\cot B+z\cot C)$. ... 25

9091. (R. W. D. Christie.)—Arrange the first sixteen natural numbers in the form of a determinant of the fourth order or degree, the sum of whose rows, columns, and diagonals are all equal..... 201

9093. (Samuel Roberts, F.R.S.)—Four spheres are bisected by the faces of a tetrahedron, and each of them passes through the three vertices on the corresponding face. Show that the sphere cutting these four spheres orthogonally has for its centre the isogonal conjugate of the centre of the circumscribing sphere (*i.e.*, if $\alpha, \beta, \gamma, \delta$ are the quadruplanar coordinates of the centre of the circumscribing sphere, $1/\alpha, 1/\beta, 1/\gamma, 1/\delta$ are those of that of the orthogonal sphere), and that the sphere

through the centres of the four spheres first mentioned has its centre at the middle point between those of the orthogonal and circumscribing spheres, and passes through the common section of those spheres. Compare the result with an analogous property of the nine-point circle in plane space 37, 178

9095. (R. Holmes, B.A.) — On a given semicircle ACB, take any point Q, and draw QX perpendicular to AB; then construct a triangle on AB with its vertex in ACB such that the in-centre is on QX. ... 44

9096. (M. F. J. Mann, B.A.)—If $n = 2^K \lambda^\alpha \mu^\beta \ldots$, and N does not contain as a factor a power of a number a higher power of which is a factor of n, prove that $N^{mL+1} - N$ is a multiple of n, where L = L. C. M. of 2^{K-2}, $\lambda^{\alpha-1}$, $\mu^{\beta-1}$,... $(\lambda-1)$, $(\mu-1)$..., and m is any integer. 52

9098. (W. J. C. Sharp, M.A.) — If a tetrahedron be such that the sums of the squares of its opposite edges are all equal, show that the nine-point circles of the triangles, which are its faces, are all sections of the same sphere. Also show, in space of n dimensions, that if (r, s) denote the line joining the r^{th} and s^{th} vertices of a simplicissimum, and if $A_1, A_2 \ldots A_{n+1}$ be $(n+1)$ magnitudes, such that $(r, s)^2 = A_r + A_s$ for all the values of r and s, the nine-point circles of all the triangles, which can be formed so as to have their vertices at three of those of the simplicissimum (see Quest. 8242), lie upon the same hyper-sphere... 165, 168

9098. (For Enunciation, see Question 3228.) 167

9100. (W. J. Barton, M.A.)—Prove that the series

$$1 + \frac{1}{n} \cdot \frac{1}{3} + \frac{3n+1}{n(n+1)} \cdot \frac{1}{3^2} + \frac{(3n+1)(3n+4)}{n(n+1)(n+2)} \cdot \frac{1}{3^3} + \ldots\ldots$$
$$+ \frac{(3n+1)(3n+4)(3n+7)\ldots(6n-5)}{n(n+1)(n+2)\ldots(2n-1)} \cdot \frac{1}{3^n}$$
$$= \frac{(3n+1)(3n+4)\ldots(6n-2)}{n(n+1)\ldots(2n-1)} \cdot \frac{1}{3^n} \ldots\ldots 41$$

9101. (W. S. B. Woolhouse, F.R.A.S.) — Let a hyperbola and an ellipse have their planes at right angles, and be such that the vertices of each shall be the foci of the other. Then if, as poles, any two points be taken, one in each branch of the hyperbola, and from them radii be drawn to the ellipse, the sum of each pair of radii will be constant for all points in the ellipse. If the two arbitrary points be taken in the same branch of the hyperbola, the difference of the radii will then be constant in value. It will also follow that, if any two poles be taken in the periphery of the ellipse, and from them radii drawn to the hyperbola, the difference of the radii will again be constant for all points in the hyperbola. 45

9102. (H. L. Orchard, B.Sc., M.A.)—Show that the series
$1^7 + 2^7 + 3^7 + 4^7 + \ldots + 9^7$ is divisible by 27..... 49

9103. (Asparagus.)—Prove that the envelope of the hyperbolas

$$\frac{x^2}{2-\sin\theta} - \frac{y^2}{2+\sin\theta} = a^2 \sin\theta$$

(θ being the parameter) is a four-cusped hypocycloid. 111

9104. (Prof. Mahendra Nath Ray, M.A., LL.B.)—Show that

$$\frac{\cos\theta}{1-\sin\theta}+\frac{\cos\phi}{1-\sin\phi}=\frac{2(\sin\theta-\sin\phi)}{\sin(\theta-\phi)+\cos\theta-\cos\phi}. \quad \ldots\ldots\ 52$$

9105. (A. Gordon.)—Required (1) the tetrahedral coordinates of the centre of the sphere circumscribing the tetrahedron of reference, (2) the conditions amongst the sides that the centre of the circum-sphere may coincide with the centre of gravity. 64

9106. (W. J. Greenstreet, B.A.)—There are 64 competitors for the Yorkshire Challenge Cup (at Football). They are drawn in pairs, and the winners of first round are drawn in pairs, and so on. Prove that Hull's chance of playing Bradford (1) in the final is $\frac{1}{2016}$, and (2) in the course of the contest is $\frac{1}{32}$. 26

9107. (F. Morley, B.A.)—Conics touch four fixed lines: show that (1) the points where they cut at right angles lie on a circular cubic through the six intersections of the lines and the three feet of perpendiculars of the diagonal triangle; and (2) the real asymptote of the cubic is parallel to the line that goes through the mid-points of the three diagonals. 89

9109. (R. Knowles, B.A.)—Tangents are drawn from a point T (hk) to meet the rectangular hyperbola $xy = a^2$ in P, Q; prove that (1) the equation to the circle TPQ is

$$2hk(x^2+y^2)-(h^3+h^2k+4a^2h)x-(h^3+hk^2+4a^2k)y+4a^2(h^2+k^2)=0;$$

and (2) if the circle meet the hyperbola again in CD, the perpendicular from the centre of the hyperbola on PQ bisects CD, and passes through its pole with respect to the hyperbola. 66

9111. (Professor Hanumanta Rau, B.A.)—Solve in *positive* integers the equation $x^2-19y^2=81$ .....(1) 48

9112. (Professor Sylvester, F.R.S.)—Prove that a number may be represented by a sequence of numbers (a single number by itself counting as a sequence) in as many ways as it contains odd factors. (*Ex.*—The odd factors of 20 are two in number, 1 and 5; and 20 is equal to the following sequences, viz., $2+3+4+5+6$, and 20.) Again, the odd factors of 45 are six in number, viz., 1, 3, 5, 9, 15, 45, and 45 may be represented by the six sequences (and no others),

$$1+2+3+4+5+6+7+8+9,\quad 5+6+7+8+9+10,\quad 7+8+9+10+11,$$
$$14+15+16,\quad 22+23,\quad 45. \ldots\ldots 48$$

9113. (Professor Haughton, F.R.S.)—The most recently found value of the Sun's distance depends upon Prof. Newcomb's determination of the velocity of light. This velocity Professor Newcomb values at 299,860 kilometres per second, with a probable error of 30 kilometres either way. Combining this with Nyrén's value of the constant of aberration 20″·492, you are required to find the solar parallax and the distance of the Sun in kilometres and British miles. 71

9114. (Professor Neuberg.) — Etant donné un triangle quelconque ABC, on mène par un même point O trois droites OA′, OB′, OC′ égales à BC, CA, AB et faisant avec un axe OX les angles XOA′ = A, XOB′ = B, XOC′ = C. Démontrer: (1) que les points O, A′, B′, C′ sont sur une circonférence égale à la circonférence ABC; (2) que les côtés du triangle A′B′C′ sont doubles des distances entre les milieux des côtés de ABC et

les pieds des hauteurs correspondantes; (3) que la droite joignant O au centre de gravité de ABC fait avec OX un angle complémentaire de l'angle de Brocard de ABC; (4) étant donné le triangle A'B'C', il existe trois triangles correspondants ABC et les trois points correspondants O sont les sommets d'un triangle équilatéral. 39

9120. (Professor Kalipada Basu, M.A.)—If p and q are integers, show that $1-\frac{q}{p+1}+\frac{q(q-1)}{(p+1)(p+2)}-\frac{q(q-1)(q-2)}{(p+1)(p+2)(p+3)}+\ldots$

$$+(-1)^q\frac{q(q-1)(q-2)\ldots 2.1}{(p+1)(p+2)\ldots(p+q)}=\frac{p}{p+q}.$$ 174

9125. (R. Tucker, M.A.)—AD, BE, CF, the altitudes of a triangle, intersect in H, and on them are taken $Ad = HD$, $Be = HE$, $Cf = HF$. Prove that, if λ_1, λ_2, λ_3 are the mid-points of *ef*, *fd*, *de*, then $A\lambda_1$, $B\lambda_2$, $C\lambda_3$ cointersect in the circum-centre; and find the area of *def*. 60

9127. (Professor Hanumanta Rau, M.A.)—Find a point in a given straight line such that the tangents from it to two given circles may be equally inclined to the given straight line. 112

9131. (W. J. Barton, M.A.)—If tangents at P and Q to a parabola meet at T, SQ meets the diameter through P and O, PQ meets OT in L, and OM perpendicular to OT in M; show that $LV.MQ = PV.LQ$, V being middle point of PQ. 122

9140. (Emile Vigarié.)—Si R, R_1, R_2 désignent respectivement les rayons du cercle circonscrit, du premier cercle de Lemoine (*triplicate ratio circle*) et le deuxième cercle de Lemoine (*cosine circle*); démontrer la relation $R^2 = 4R_1^2 - R_2^2$. 99

9142. (R. W. D. Christie. See Question 8700.)—If

$$\Sigma_r = 1^r + 2^r + 3^r \ldots\ldots n^r,$$

prove that $(9\Sigma_{11}+30\Sigma_9+9\Sigma_7)/\Sigma_3 = (11\Sigma_{10}+30\Sigma_8+7\Sigma_6)/\Sigma_2$. 84

9145. (A. R. Johnson, M.A.)—(1) From a point P (x, y) normals PN_1, PN_2, PN_3, PN_4 are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$; show that

$$\Delta N_2N_3N_4 . \Delta N_3N_4N_1 . \Delta N_4N_1N_2 . \Delta N_1N_2N_3$$
$$= a^4b^4c^{-12}\left\{(c^4-a^2x^2-b^2y^2)^3-27a^2b^2c^4x^2y^2\right\}.$$

(2) If two conics cut at the points P_1, P_2, P_3, P_4, show that $\Delta P_2P_3P_4 . \Delta P_3P_4P_1 . \Delta P_4P_1P_2 . \Delta P_1P_2P_3 = T/\left\{(.12.)^2-4.1^2.2^2.\right\}^3$, where T is the tact-invariant (Salmon, p. 325, 5th ed.), and $.1^2.$, $.2^2.$ are the discriminants of the two pairs of lines drawn through any point parallel to the asymptotes, and $.12. = 0$ is the condition that the two pairs should be harmonically conjugate. 63

9147. (S. Constable.)—The vertex of a triangle is fixed, and the vertical angle given, while the base angles move on two fixed lines. Perpendiculars are let fall from the vertex on the two fixed lines, and the feet of these perpendiculars are joined to the mid-points of the sides of the triangle; prove that the locus of the point where these lines intersect is a circle. 122

9155. (Professor Crofton, F.R.S.)—Find an operator ζ which satisfies the condition $\zeta\chi - \chi\zeta \equiv 2D$, where χ is a given function of x, and $D = d/dx$. 53

9156. (Professor Neuberg.)—Soient a, b, c, d les hauteurs d'un tétraèdre ABCD; $\alpha, \beta, \gamma, \delta$ les angles des faces avec un plan P; p la distance du centre de la sphère ABCD à P. Démontrer que tout point M de P satisfait à la relation

$$\frac{\cos\alpha}{a}(AM)^2 + \frac{\cos\beta}{b}(BM)^2 + \frac{\cos\gamma}{c}(CM)^2 + \frac{\cos\delta}{d}(DM)^2 = 2p \quad \ldots\ldots(1).$$

..... 91

9158. (Professor Munindranath Bhattacharyya.)—Show that the equation to the n^{th} pedal of the curve $r = f(p)$ is $\frac{r^{n+1}}{p^n} = f\left(\frac{r^n}{p^{n-1}}\right)$. 94

9159. (Professor Wolstenholme, M.A., Sc.D.)—Any point P of a certain curve being taken, the corresponding points of an involute and of the evolute are Q, Q'; prove that, if QQ' pass always through a fixed point O, the curve will have for its intrinsic equation $s = a\sin m\psi$ or $s = Ae^{m\psi} + Be^{-m\psi}$; also OQ : OQ' will be a constant ratio. 95

9161. (Professor Hudson, M.A.)—If the semi-axes of an ellipse are CA, CB; prove that the major axes of all ellipses touching the given ellipse and having their foci at the ends of a diameter thereof are equal and equal to 2AB. 96

9168. (The Editor.)—Show that one root of the equations

$$x^2 + 10x = 5(y^3 - 7x^3) = 20x^2,\ y + z = 28,$$

is $x = 4{\cdot}510664$, and ascertain the nature of the other roots. 53

9170. (Samuel Roberts, F.R.S.)—Show that, (1) if the sides of a triangle (produced if necessary) are cut by a circle, and on the linear intercepts as diameters three circles are drawn, the radical centre of these is the "isogonal conjugate," relative to the triangle, of the centre of the first-named circle; (2) if the planes of the faces of a tetrahedron are cut by a sphere, and on the circular intercepts as diametral sections four spheres are constructed, the radical centre of these spheres is the "isogonal conjugate" relative to the tetrahedron of the centre of the first-named sphere. ["Isogonal conjugates" are points whose trilinear or quadruplanar coordinates are proportional to (α, β, γ), $(1/\alpha, 1/\beta, 1/\gamma)$, or $(\alpha, \beta, \gamma, \delta)$, $(1/\alpha, 1/\beta, 1/\gamma, 1/\delta)$. These theorems may be obtained geometrically.] 37, 179

9171. (H. L. Orchard, B.Sc., M.A.)—Show, by an ordinary quadratic method, that the roots of the equation

$$u^4 + 11u^3 + 44u^2 + 132u + 144 = 0$$

are $u = \frac{1}{2}\{[22(41)^{\frac{1}{2}} - 30]^{\frac{1}{2}} - [(41)^{\frac{1}{2}} + 11]\} = -1{\cdot}71843$. 76

9178. (D. O. S. Davies, M.A.)—If P is any point on a rectangular hyperbola whose centre is C, prove that the envelope of the circles whose diameters are CP, is the inverse of the hyperbola with regard to its auxiliary circle. 123

9180. (Charlotte A. Scott, B.Sc.) — One end of a string is fixed above a uniformly flowing stream; the other end is attached to the middle point of the end of a plank which is partially immersed in

the water, with its plane opposed to the stream: obtain equations for determining the position of equilibrium, and show that the string and plank cannot be in the same straight line. 71

9181. (Sarah Marks, B.Sc.)—The mid-points of the perpendiculars drawn from A, B, C on the opposite sides of the triangle ABC are P, Q, R; also D, E, F are the mid-points of BC, CA, AB; prove that (1) PQ, DE and the fourth harmonic to CB, CR, CA meet in a point; and (2) the three such points lie on a straight line..... 121

9184. (Emile Vigarié. Suggested by Question 9140.) — Si le segment OK qui joint le centre du cercle circonscrit au point de Lemoine (*Symmedian point*) est divisé dans le rapport $p : q$ par le centre d'un cercle de Tucker, et si R_1, R_2 sont respectivement les rayons du premier cercle de Lemoine et du deuxième cercle de Lemoine, le rayon P du cercle de Tucker considéré sera donné par la formule

$$P = [4q^2R_1^2 + (p^2 - q^2)\,R_2^2]^{\frac{1}{2}} / (p+q). \ldots\ldots 99$$

9185. (Major-General P. O'Connell.) Mr. Thomas Weddle, by a solution extending over six printed pages, has found one root of $1379664x^{622} + 2686034 \times 10^{432} \times x^{153} - 17290224 \times 10^{578} \times x^{60} + 2524156 \times 10^{574} = 0$ to be $x = 8{\cdot}367975431$; it is required to find, by using a seven-figure table of common logarithms, another positive root containing seven figures, the solution occupying not more than one octavo page. 128

9187. (J. White.)—Prove that $8 \cos \frac{2}{7}\pi \cos \frac{4}{7}\pi \cos \frac{1}{7}\pi = 1$. 124

9197. (Professor Haughton, F.R.S.)—If 2E denote the total sun-heat received by the whole Earth in a year, and if Δ denote the obliquity of the ecliptic, prove that each hemisphere receives, in a year,

$$\text{in summer, } E \times \frac{\pi + 2\sin\Delta}{2\pi}; \quad \text{and in winter, } E \times \frac{\pi - 2\sin\Delta}{2\pi}. \quad 85$$

9198. (Professor Crofton, F.R.S.)—Prove that

$$e^{ia(x+D)^2}\, e^{ibx^2} = (1-a-ab)^{-\frac{1}{2}}\, e^{i[(a+b+ab)/(1-a-ab)]\frac{1}{2}x^2}. \ldots\ldots 69$$

9199. (Professor Hudson, M.A.)—Given an angle of a triangle, the direction of the sides containing it, and the circum-radius; find the locus of the orthocentre. 90

9201. (Professor Matz, M.A.)—P is a point within a triangle ABC, and AP, BP, CP meet the opposite sides in A′, B′, C′, respectively; if Pa, Pb, Pc be measured along PA, PB, PC so that these last are harmonic means between PA′, Pa; PB′, Pb; PC′, Pc, respectively, prove that a, b, c lie on a straight line 95

9202. (Professor Ignacio Beyens, M.A.)—Si R_1, R_2, ρ sont les rayons du premier cercle de Lemoine, du deuxième cercle de Lemoine, et du cercle de Brocard, on aura: $\rho^2 = R_1^2 - R_2^2$. 99

9206. (Professor Schoute.)—Given the triangle ABC; prove that there are three tangents A′BC, AB′C, ABC′ having in common with ABC two vertices and the Lemoine point; and that the new triangle A′B′C′ is in perspective with ABC..... 104

9207. (Professor Bordage.)—If a and b are perfect numbers (*nombres entiers*), $a^2b\,(a^3 + 2a^2 - a - 2)(a^4 - b^4)$ is divisible by 720. 93

9209. (Professor Mahendra Nath Ray, M.A., LL.B.) — Show that the value of $\dfrac{6+4x}{3x^2} - \dfrac{1}{x - \log_e(1+x)}$, when $x = 0$, is $\frac{1}{3}$. 108

9211. (The Editor.)—A straight rod, divided into $2n$ equal parts, is broken at random into three parts at the points of division; find the probability that these three parts will form (1) a triangle of any kind, (2) an acute-angled triangle. 90

9214. (Rev. T. C. Simmons, M.A. Connected with Quest. 9099.)—Show how to construct a system of triangles having one given point for circumcentre and another given point for orthocentre, and having given either the product of the cosines of the angles (not $> \frac{1}{8}$), or the sum of the squares of the sides. Show, moreover, that the envelope of the sides of any such system of triangles is an ellipse having the two given points for foci..... 110

9218. (A. Gordon.)—Show that, if lines are drawn perpendicular to and through the middle points of the sides of a scalene triangle, outwards, and distances measured along these lines proportional to the corresponding sides, there is only one set of points so found which can form an equilateral triangle. 105

9222. (Sarah Marks, B.Sc.)—If a, b, c be positive quantities which are not all equal to each other, prove that

$$(b+c-a)(b-c)^2+(c+a-b)(c-a)^2+(a+b-c)(a-b)^2$$

will be a positive quantity. 113

9223. (H. L. Orchard, B.Sc., M.A.)—Two crews, rowing the one 18 miles with the current, and the other 21 miles against it, pass one another, and complete their courses by rowing for 1 and 5 hours more, respectively. Supposing the crews to start at the same time and to row uniformly, find their rates of speed. 202

9225. (R. W. D. Christie.) — Show that the sum of the primes $= 30\Sigma(\Sigma+1)+18-C$; where Σ represents the number of the set (of four) to which the given prime belongs, and C = sum of multiples of primes. [*Ex. gr.*, To find the sum of primes to 97 inclusive,

$$30\Sigma(\Sigma+1)=30\times 6\times 7=1260, \quad C=7(7+11+13)=217;$$

hence the required sum is $1260+18-217=1061$.] 202

9226. (J. White.)—Prove that
$1^3+2^3+3^3\ldots M^3$ is a factor of $(1^5+2^5+3^5\ldots M^5)\times 3$. 116

9228. (Rev. T. P. Kirkman, F.R.S.)—If two uniform rods AC, BD, whose ends A, B rest on a smooth horizontal plane, are connected by a hinge at D, and tied together with a string AB, find the tension of AB. 70

9231. (Professor Crofton, F.R.S.)—Prove that, if ξ be any operator containing only x, D, and χ any function of x,

$$\xi\chi=\chi\xi+\chi'\frac{d\xi}{d\mathrm{D}}+\frac{\chi''}{1\,.\,2}\frac{d^2\xi}{d\mathrm{D}^2}+\&c.,$$

χ', $\chi''\ldots$ being derived.

[The theorem may be otherwise expressed as follows:—

$$\xi f(x)=f\left(x+\frac{d}{d\mathrm{D}}\right)\xi].$$ 88

9232. (Professor Hudson, M.A.)—If the bisector of an angle of a triangle pass through the centre of the nine-points circle, prove that that angle must be 60°..... 113

9237. (Professor Nilkantha Sarkar, M.A.)—Find three plane right-angled triangles whose areas are equal. 118

9238. (Professor Hanumanta Rau, M.A.)—If AA′, BB′, CC′ are equal tangents drawn in the same sense to the circum-circle of a triangle ABC; prove that A′B′ : B′C′ : C′A′ = AB : BC : CA. 102

9240. (Professor Nilkantha Sarkar, M.A.)—If a triangle be inscribed in a circle of radius a, prove that its mean area is $3a^2/2\pi$. 91

9241. (Professor Wolstenholme, M.A., Sc.D.) — 1. The focus of a given parabola is S, and P is any point on the parabola; with focus S and vertex P is described another parabola meeting the former again in one real point Q; prove that PQ will be normal at P to the first parabola; and find the locus of the polar of PQ with respect to the second parabola.

2. Prove by elementary geometry the following property of the circle, which is the reciprocal of the above theorem with regard to S. O is a fixed point on a given circle, OY perpendicular on the tangent to this circle at a point P; another circle is described on OY as diameter, and the second common tangent to the two circles meets the former tangent in R; prove that the angle POR is a right angle. 86

9242. (Professor Schoute.)—The triangle ABC is the intersection of the plane δ with the faces of the trihedron $\alpha\beta\gamma$. To find the locus of the centroid of ABC, when δ moves about (1) a fixed line, (2) a fixed point. 146

9243. (Professor Nilkantha Sarkar, M.A.)—Show that the asymptotes of an equilateral hyperbola are the loci of the extremities of lengths measured on the tangents, equal in length to the normals, starting from the point of contact. 114

9244. (Professor Munindranath Bhattacharyya, M.A.) — Professor Wolstenholme obtains very elegantly by parts the integrals of $e^{mx}\cos nx$ and $e^{mx}\sin nx$. Show that the integrals can be obtained by a different process. [Gregory, in his *Methods of Diff. and Integ. Calc.*, suggests that these two integrals are obtainable by "Double Integration by Parts."] 110

9245. (Professor Ignacio Beyens.)—Démontrer par la méthode des equipollences la proposition bien connue : Si A_1, A_2, A_3 sont trois points pris sur le côté OX d'un angle XOY, et B_1, B_2, B_3 sont pris sur le côté OY, les points de rencontre M de A_2B_3, A_3B_2; N de A_3B_1, A_1B_3; P de A_1B_2, A_2B_1 sont en ligne droite. 87

9246. (Professor Bordage.)—Solve the equation

$$16^{\cos^2 x + 2\sin^2 x} + 4^{2\cos^2 x} = 40. \ldots\ldots 107$$

9249. (D. Edwardes.)—The normal at a point P of an ellipse meets the curve again in Q. Find the maximum value of the angle CQP, C being the centre of the ellipse. 147

9251. (S. Tebay, B.A.) — In the tetrahedron VABC, the dihedral angles over the base ABC are X, Y, Z. If perpendiculars from A, B, C on the faces make angles X′, Y′, Z′ with VA, VB, VC, prove that

$$\sin X \cos X' = \sin Y \cos Y' = \sin Z \cos Z' \ldots\ldots 115$$

9252. (R. W. D. Christie.)—Prove that the series

$$[1 + 3^2 + 3^4 \ldots\ldots 3^{2(n-1)}]$$

equals a sequence from unity $(1 + 2 + 3 \ldots\ldots n)$ 116

9254. (W. J. C. Sharp, M.A.)—If any point P in its plane be joined to the vertices of a triangle ABC, and if R_1, R_2, R_3 be the radii of the circles described about BPC, CPA, APB, respectively; then, a, b, c being the sides of the triangle,

$$\frac{a^2 \cdot b^2 \cdot c^2}{R_1^2 \cdot R_2^2 \cdot R_3^2} = \left(\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3}\right)\left(-\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3}\right)$$
$$\left(\frac{a}{R_1} - \frac{b}{R_2} + \frac{c}{R_3}\right)\left(\frac{a}{R_1} + \frac{b}{R_2} - \frac{c}{R_3}\right). \quad \ldots\ldots\ldots 103$$

9261. (Professor Nash, M.A.)—The general equation of a parabola in trilinear coordinates is

$$(l\alpha + m\beta + n\gamma)^2 = (a\alpha + b\beta + c\gamma)(l'\alpha + m'\beta + n'\gamma) \text{ or } L^2 = SL';$$

if $$A = \Sigma l^2 - 2\Sigma mn \cos A, \quad A' = \Sigma l'^2 - 2\Sigma m'n' \cos A,$$
$$B = \Sigma ll' - \Sigma (mn' + m'n) \cos A,$$

prove that the equation of the axis is $2AL = BS$, the focus is the intersection of this line with $4AL' = A'S$, the equation of the directrix is $4BL - 4AL' = A'S$, and the Latus rectum $= 2\Delta (AA' - B^2)^{\frac{1}{2}} / A^{\frac{3}{2}}A$.. 148

9262. (Professor Neuberg.)—Soit O′ le conjugué isogonal (ou l'inverse) du centre de la sphère circonscrite à un tétraèdre ABCD. Démontrer que ce point est le centre de la sphère inscrite au tétraèdre, qui a pour sommets ses projections sur les faces du tétraèdre ABCD. 103

9263. (Professor Wolstenholme, Sc.D.)—Prove that, if

$$\alpha + \beta + \gamma + \delta = \pi \text{ (or any odd multiple of } \pi),$$

$$4(\sin\beta\sin\gamma + \sin\alpha\sin\delta)(\sin\gamma\sin\alpha + \sin\beta\sin\delta)(\sin\alpha\sin\beta + \sin\gamma\sin\delta)$$
$$= 4(\cos\beta\cos\gamma + \cos\alpha\cos\delta)(\cos\gamma\cos\alpha + \cos\beta\cos\delta)(\cos\alpha\cos\beta + \cos\gamma\cos\delta)$$
$$= (-\sin\alpha + \sin\beta + \sin\gamma + \sin\delta)(-\sin\beta + \sin\gamma + \sin\delta + \sin\alpha)(\ldots\ldots)(\ldots\ldots)$$
$$= (-\cos\alpha + \cos\beta + \cos\gamma + \cos\delta)(-\cos\beta + \cos\gamma + \cos\delta + \cos\alpha)(\ldots\ldots)(\ldots\ldots)$$
$$= (\sin\alpha\cos\alpha + \sin\beta\cos\beta + \sin\gamma\cos\gamma + \sin\delta\cos\delta)^2$$
$$= 4\sin(\beta+\gamma)\sin(\alpha+\delta)\sin(\gamma+\alpha)\sin(\beta+\delta)\sin(\alpha+\beta)\sin(\gamma+\delta). \ldots 148$$

9265. (Professor Mahendra Nath Ray, M.A., LL.B.)—If

$$\tan x_1 \tan x_2 \tan x_3 \ldots\ldots \tan x_n = \tan A,$$

prove that $$\cos 2A = \frac{s_1 + s_3 + s_5 + \ldots}{1 + s_2 + s_4 + \ldots},$$

where s_r = the sum of the product of the quantities $\cos 2x_1$, $\cos 2x_2$, ... $\cos 2x_n$ taken r at a time..... 149

9270. (Professor Satis Chandra Ray, M.A.)—If p_1, p_2, p_3, p_4, p_5, p_6 be the central perpendiculars on the tangent planes at the six points where the normals through a given point (X, Y, Z) meet the ellipsoid, and ρ_1, ρ_2, ρ_3, ρ_4, ρ_5, ρ_6 be the normal distances of the point from the ellipsoid, show that

(1) $\Sigma p_1 \rho_1 = 0$, (2) $p_1\rho_1 p_2\rho_2 \ldots p_6\rho_6 = a^4b^4c^4\left\{1 - X^2/a^2 - Y^2/b^2 - Z^2/c^2\right\}$.
..... 150

9273. (Professor Bordage.)—Solve the equation

$$7346 \times 7^{\sec x} + 7^{1+\sec x} - 7010 \times 7^{2\sec x} - 7^{3+2\sec x} + 3 \times 7^{2+3\sec x} = 147 \ldots. 123$$

9275. (Captain H. Brocard.)—Les pieds des hauteurs d'un triangle sont les foyers de paraboles tangentes à deux côtés et aux deux hauteurs

correspondantes. Ces paraboles admettent pour directrices, pour paramètres et pour sommets, respectivement, les côtés et les hauteurs du triangle orthocentrique et les milieux des hauteurs de ce triangle. Donner l'expression analytique des paramètres de ces paraboles. 112

9276. (E. Vigarié.) — Dans tout quadrilatère harmonique, (1) la somme des inverses des carrés des côtés est égale à deux fois l'inverse de la puissance du point de Lemoine par rapport au cercle circonscrit; (2) le produit des côtés est égal à quatre fois le produit des droites qui joignent un sommet au milieu de la diagonale qui ne passe pas par ce sommet. 119

9278. (R. W. D. Christie.)—Show that it is always possible to arrange the first (n^2) natural numbers in the form of a determinant the sum of whose rows, columns, and diagonals is $\frac{1}{2}n(n^2+1)$ when $n>2$.

Example.

1,	20,	33,	34,	17,	6	
32,	8,	30,	7,	11,	23	
24,	35,	15,	16,	2,	19	
18,	10,	21,	22,	27,	13	
5,	26,	9,	28,	29,	14	
31,	12,	3,	4,	25,	36	 203

9280. (Major-General P. O'Connell.)—A reservoir full of water is of the form of an inverted cone. It is required to find an expression for the time in which its surface will fall through a given depth, if water is being drawn off at a uniform rate through sluices, *i.e.*, the same volume in each successive second of time, while a uniform depth per second is being removed by evaporation. 102

9283. (R. Knowles, B.A. Connected with Quest. 9109.) — A circle passes through two points P, Q on a rectangular hyperbola, and meets it again in C, D; prove that the perpendicular from the centre of the hyperbola on PQ bisects CD and passes through its pole with respect to the hyperbola, and *vice versâ*. 150

9284. (F. R. J. Hervey.) — Prove that the locus of the feet of the normals drawn from the centre of a circle to the variable conic passing through four points given on the circle is a rectangular hyperbola.... 151

9286. (H. L. Orchard, B.Sc., M.A.)—Jack and Jill go a journey of eighty miles; the one on a tricycle, the other on a bicycle. Jack starts first, and, at the end of four hours, is overtaken by Jill. Jill completes the course in 2¼ hours, and Jack completes it in the same time which Jill took to overtake him. Assuming Jack and Jill to travel uniformly, find their speeds. 114

9287. (J. White.)—If D be the circumcentre and E the incentre of the triangle ABC, prove that (1) if DE passes through the orthocentre,

$$\cos^2 A \cos B + \cos^2 B \cos C + \cos^2 C \cos A = \cos^2 A \cos C + \cos^2 B \cos A + \cos^2 C \cos B,$$

and (2) if this condition is satisfied, the incentre, the circumcentre, and the centre of the nine-points circle of the triangle are collinear. 124

9289. (R. Holmes, B.A.)—From the equations

$$C_{s-1}\cos ml_{s-1} - D_{s-1}\sin ml_{s-1} = C_s,$$

$$(C_{s-1}\sin ml_{s-1} + D_{s-1}\cos ml_{s-1})\,\omega_{s-1} = (C_s \sin ml_s + D_s \cos ml_s)\,\omega_s,$$

determine the ratio of C_s and D_s to D_1, being given that $C_1 = 0$; and hence prove the relation

$$\frac{D_1\omega_1 \cos ml_1}{D_s\omega_s \cos ml_s} = 1 + \frac{C_s}{D_s} \tan ml_s. \quad \ldots\ldots\ldots\ldots\ldots 151$$

9291. (J. O'Regan.) — Standing at an unknown distance from an octagonal house in the line perpendicular to one of its sides, I observed the angle subtended by the extreme visible corners; then, advancing 100 feet nearer in the same line, I found the same angle to be triple of what it was at the first station; and, advancing 20 feet still nearer, found I was in a line with two of the sides. Determine a side of the octagon. 152

9292. (J. L. Bury.)—A and B have a piece of work to do. B commences first and works for two-thirds of the time that it would take A to do the whole; then A sets to work and finishes it. Now, both working together could have done it two hours sooner, and B would then have done only half the quantity of work he left A to do. Show that A and B could each have done the work separately in three and six hours respectively. 153

9295. (Emily Perrin, B.Sc.)—ABC is a triangle, O the centre of its inscribed circle, and A_1, B_1, C_1 the centres of the circles escribed to the sides BC, CA, AB respectively; L, M, N the points where these sides are cut by the bisectors of the angles A, B, C. Show that the orthocentres of the three triangles LB_1C_1, MC_1A_1, NA_1B_1 form a triangle similar and similarly situated to $A_1B_1C_1$, and having its orthocentre at O. 153

9298. (The Editor.)—If α be the semi-vertical angle of a cone, p the perpendicular from the vertex on a plane which cuts a complete elliptic section from the cone, and θ the inclination of the axis of the cone to the cutting plane, show that the area (Σ) of the elliptic section is

$$\Sigma = \frac{\pi p^2 \sin^2\alpha \cos\alpha}{\sin^{\frac{3}{2}}(\theta+\alpha)\sin^{\frac{3}{2}}(\theta-\alpha)} = \frac{2^{\frac{3}{2}}\pi p^2 \sin^2\alpha \cos\alpha}{(\cos 2\alpha - \cos 2\theta)^{\frac{3}{2}}}. \quad \ldots\ldots\ldots 153$$

9323. (Emile Vigarié.)—Le point de Lemoine d'un triangle ABC, son point anticomplémentaire (point de Lemoine du triangle obtenu en menant par les sommets de ABC des parallèles aux côtés opposés), et les points réciproques (*isotomic points*) des points de Brocard sont les sommets d'un parallélogramme. 154

9332. (G. S. Carr, M.A.)—If H be the mid-point of either diagonal of an in-quadrilateral of a circle, and if the sides produced meet in E and F, the two diagonals are proportional to the diagonals of the parallelogram on EH, HF..... 155

9333. (G. H. Bryan, B.A.) — The two diagonals and two opposite sides of a square are formed of four rigid bars freely jointed at their extremities. One diagonal is fixed and the other bars are moved about in their plane. Show that the middle point of the other diagonal describes a lemniscate. 156

9335. (The Editor.) — If from a point Q on a circle AQX, a perpendicular QR be drawn to the diameter AX, and AP be drawn to touch at P a circle with QR as diameter and O as centre, and PG be a perpendicular on QR, prove that (RG : GQ) = 4 (AR : RX). 156

9342. (Asparagus.)—The fourth common tangent to an ellipse and

its circle of curvature at a point P touches the ellipse in Q; prove that, the coordinates of P being $(a\cos\theta, b\sin\theta)$, the equation of PQ is

$$x/a\cos\theta\,(1-2e^2\cos^2\theta+e^2\cos^4\theta)-y/b\sin\theta\,(1-e^2\cos^4\theta)$$
$$+1-2\cos^2\theta+e^2\cos^4\theta=0 \ldots\ldots 156$$

9343. (For Enunciation, see Question 6899.)

9379. (R. W. D. Christie.)—An annuity of £20 a year for 21 years is sold for £220; what rate per cent. did he get for his money? 132

APPENDIX I.

A Brief Explanation of the Advantages to be derived from using the Aid to Approximate Calculation. (By D. Biddle.) 125

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Solutions of Unsolved Questions. (By R. F. Davis, M.A.)..... 137

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APPENDIX IV.

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APPENDIX V.

Solutions of some Old Questions. (By R. W. D. Christie.)..... 192

CORRIGENDA.

Page 72, line 14, *for* 8922 *read* 6899 and 9343.

Page 79, last line, *for* "l'équation (1)" *read* "la formule (2)."

Page 153, line 25, *for* $AB'.AC'$ *read* $AB_1.AC_1$.

Page 186, Art. x., *add*:—"The ninth and tenth perfect numbers appear to be $2^{40}(2^{41}-1)$ and $2^{46}(2^{47}-1)$ respectively. As there appears to be no tolerably easy test for large primes, it is difficult to obtain others."

MATHEMATICS

FROM

THE EDUCATIONAL TIMES.

WITH ADDITIONAL PAPERS AND SOLUTIONS.

9072. (Ch. Hermite, LL.D., Membre de l'Institut.)—On demande de prouver que l'équation transcendante

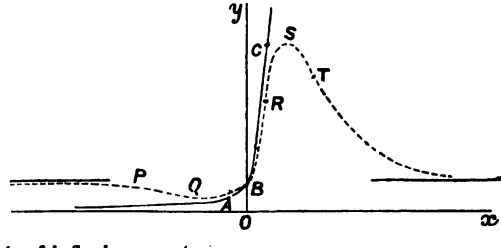
$$e^{2x} = \{(x+a+1)^2+1\} / \{(x+a-1)^2+1\},$$

où a désigne une constante réelle quelconque, n'admet qu'une seule et unique racine réelle de même signe que a, et moindre en valeur absolue que le logarithme népérien de $\sqrt{2}+1$.

Solutions by (1) A. R. Johnson, M.A.; (2) D. Edwardes.

1. In the figure, the dotted curve has for its equation

$$y = [(x-1)^2+1] / [(x+1)^2+1].$$



The points of inflexion are at

$$P\,(x = -\sqrt{3}-1,\ y = 2-\sqrt{3}),\ R\,(x = \sqrt{3}-1,\ y = 2+\sqrt{3}),$$

and at $T\,(x=2,\ y=\frac{2}{5})$. The maximum and minimum ordinates occur at $S\,(y = 3+2\sqrt{2},\ x = \sqrt{2})$, and $Q\,(y = 3-2\sqrt{2},\ x = -\sqrt{2})$, respectively.

The only asymptote is $y = 1$, and there are no other real parts except those exhibited.

The continuous curve is the real part of $y = e^{2x}$. It touches the axis of x at $-\infty$, touches the dotted curve at B with contact of second order, and proceeds to infinity as in figure. For points between BC, BS having the same y, the tangents of inclination of the tangents to the axis of x are in

the ratio $\{(x-1)^2+1\}\{(x+1)^2+1\} / 2(2-x^2)$, $\{x < \sqrt{2}\}$,

so that the continuous curve is the steeper. At A $\{y = 3-2\sqrt{2}$, $x = -\log_e(\sqrt{2}+1)\}$ the ordinate is equal to the minimum ordinate of the dotted curve, and at C $\{y = 3+2\sqrt{2}$, $x = \log_e(\sqrt{2}+1)\}$ the ordinate of the continuous curve is equal to the maximum ordinate of the dotted one.

If we shift the dotted curve a distance a to the left, the coordinates of the point of intersection will satisfy $e^{2x} = [(x+a+1)^2+1] / [(x+a-1)^2+1]$. Now, starting from the position in figure, where there is only one real point ($a=0$) and that repeated thrice, we see, as the curve moves towards the right (a negative and increasing numerically), there is only one real point of intersection, which descends BA till Q comes to A, and afterwards mounts AB, finally resting at B; *i.e.*, as a increases numerically from 0 to $-\{\sqrt{2}-\log_e(\sqrt{2}+1)\}$, the only real root is negative and increases numerically from 0 to $\log_e(\sqrt{2}+1)$, and as a increases numerically from $-\{\sqrt{2}-\log_e(\sqrt{2}+1)\}$ to $-\infty$, the only real root decreases numerically from $\log_e(\sqrt{2}+1)$ to 0 still remaining negative.

Now, starting from the position in figure, let the dotted curve move to the left (a increasing positively). Then there is only one real point of intersection, which ascends BC till S comes on B, and afterwards sinks down CB, finally resting at B; *i.e.*, as a increases from 0 to $\sqrt{2}-\log_e(\sqrt{2}+1)$, there is only one real root which is positive and increases from 0 to $\log_e(\sqrt{2}+1)$, and as a increases from $\sqrt{2}-\log_e(\sqrt{2}+1)$ to ∞, the only real root is positive and decreases from $\log_e(\sqrt{2}+1)$ to zero.

Thus, summarily: "l'équation transcendante, etc.," except when $a=0$, and then the equation has a triple zero root.

2. *Otherwise*:—This appears from drawing the curves

$$y = e^{2x} \quad \ldots\ldots\ldots\ldots (1),$$

$$y = \frac{(x+a+1)^2+1}{(x+a-1)^2+1} \quad \ldots\ldots (2).$$

In (2) y always lies between $(\sqrt{2}+1)^2$ and $(\sqrt{2}-1)^2$. When

$$x = 0, \quad y = \frac{(a+1)^2+1}{(a-1)^2+1},$$

and is $\gtrless 1$ as a is $\gtrless 0$.

Asymptote $y = 1$. When $x = -a \pm \sqrt{2}$, $y = (\sqrt{2} \pm 1)^2$,

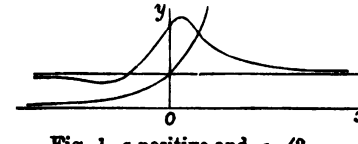


Fig. 1, a positive and $< \sqrt{2}$.

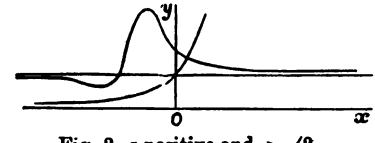


Fig. 2, a positive and $> \sqrt{2}$.

and $\frac{dy}{dx} = 0$. When $y = 1$,

$$x + a = 0, \quad \frac{dy}{dx} = 2,$$

and the curves are parallel. When $x = 0$, $\frac{d^2y}{dx^2}$ has the same sign as $2 - a^2$. The points of inflexion are given by

$$(x+a)^3 - 6(x+a) + 4 = 0,$$

or $x + a = 2, \sqrt{3} - 1, -(\sqrt{3}+1)$.

(1) intersects y-axis at the point $y = 1$. Asymptote $y = 0$.

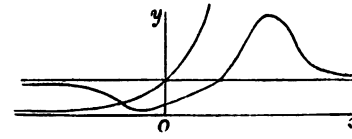


Fig. 3, a negative and $< \sqrt{2}$.

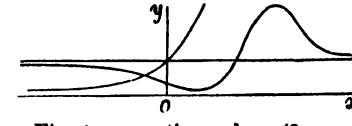


Fig. 4, a negative and $> \sqrt{2}$.

The abscissa of the point of intersection must be < the abscissa of the point for which

$$y = (\sqrt{2}+1)^2 = e^{2x}, \quad \text{viz., } x = \log(\sqrt{2}+1).$$

[The PROPOSER's solution is as follows:—

Considérons l'équation transcendante par rapport à la constante a qui y entre au second degré. On trouve, pour que cette quantité soit réelle, la condition $-e^{4x} + 6e^{2x} - 1 > 0$,

ce qui donne: $(\sqrt{2}-1)^2 < e^{2x} < (\sqrt{2}+1)^2$.

L'équation proposée ne peut donc admettre de racine supérieure à la limite $\log(\sqrt{2}+1)$; mais en l'écrivant sous cette forme:

$$e^x[(x+a+1)^2+1] - e^{-x}[(x+a-1)^2+1] = 0,$$

la dérivée est simplement: $e^x(x+a+2)^2 + e^{-x}(x+a-2)^2$;

il n'y a par conséquent qu'une seule racine, etc.]

7740. (Professor SYLVESTER, F.R.S.)—If $px^2 + qx + r = 0$ be an equation in matrices of the 2nd order, or (if any one pleases to say so) in quaternions, it follows from HAMILTON's investigations (see *Lectures on Quaternions*, pp. 631, 632) that this equation has in general 6 roots, reducible to 4 if a certain single ordinary equation is satisfied by the elements of the matrices p, q, r. Show that, if this condition is satisfied for any given system of matrices p, q, r, it will also be satisfied by any three homogeneous linear functions of p, q, r.

[Professor SYLVESTER recommends the student to prove that the condition referred to is found by equating to zero the discriminant in respect to u, v, w of the determinant to $up + vq + wr$, from which its invariability when p, q, r undergo linear substitution is an immediate consequence. It will, of course, be understood that if

$$p = \begin{matrix} a, & b \\ c, & d \end{matrix}, \quad q = \begin{matrix} a', & b' \\ c', & d' \end{matrix}, \text{ and } r = \begin{matrix} a'', & b'' \\ c'', & d'' \end{matrix}, \quad up + vq + wr \text{ means}$$

$$au + a'v + a''w, \quad bu + b'v + b''w, \quad cu + c'v + c''w, \quad du + d'v + d''w.]$$

Solution by Professor SEBASTIAN SIRCOM, M.A.

$$\text{Let } x = \begin{matrix}\alpha, & \beta,\\ \gamma, & \delta,\end{matrix} \quad \text{then} \quad x^2 = \begin{matrix}\alpha^2+\beta\gamma, & \beta(\alpha+\delta),\\ \gamma(\alpha+\delta), & \delta^2+\beta\gamma,\end{matrix}$$

$$px^2 = \begin{matrix}a(\alpha^2+\beta\gamma)+b\gamma(\alpha+\delta), & a\beta(\alpha+\delta)+b(\delta^2+\beta\gamma),\\ c(\alpha^2+\beta\gamma)+d\gamma(\alpha+\delta), & c\beta(\alpha+\delta)+d(\delta^2+\beta\gamma),\end{matrix}$$

$$qx = \begin{matrix}a'\alpha+b'\gamma, & a'\beta+b'\delta,\\ c'\alpha+d'\gamma, & c'\beta+d'\delta,\end{matrix}$$

by the rules for the composition of matrices. Addition is performed by adding corresponding constituents, and px^2+qx+r is to be equal to the matrix 0, hence each constituent in the sum is 0.

Therefore, if $\alpha+\delta = x, \quad \beta\gamma-\alpha\delta = y,$

$$a(ax+a')+\gamma(bx+b')+ay+a''=0, \quad a(cx+c')+\gamma(dx+d')+cy+c''=0,$$
$$\beta(ax+a')+\delta(bx+b')+by+b''=0, \quad \beta(cx+c')+\delta(dx+d')+dy+d''=0.$$

From these we can obtain α, β, γ, δ linearly in terms of x and y. Forming x and y from the values so obtained, we have

$$x\{(ax+a')(dx+d')-(bx+b')(cx+c')\}+(dx+d')(ay+a'')+(ax+a')(dy+d'')$$
$$-(cx+c')(by+b'')-(bx+b')(ey+c'')=0,$$
$$y\{(ax+a')(dx+d')-(bx+b')(cx+c')\}+(ay+a'')(dy+d'')$$
$$-(cy+c'')(by+b')=0.$$

Now, let ABCFGH be the constituents of the discriminant to

$$(au+a'v+a''w)(du+d'v+d''w)-(bu+b'v+b''w)(cu+c'v+c''w),$$

which is the determinant to $up+vq+wr$,

then $\quad A = ad-bc, \quad 2F = a'd''+a''d'-b'c''-b''c'$, &c.,

and the equations giving x and y may be written

$$(Ax^2+2Hx+B+2G)x+2Axy+2Hy+2F=0,$$
$$(Ax^2+2Hx+B+2G)y+Ay^2+C=0.$$

Eliminating y, we have

$$(Ax^2+2Hx)(Ax^2+2Hx+B+2G)^2+4HF(Ax^2+2Hx+B+2G)$$
$$-4AC(Ax^2+2Hx)-4CH^2-4AF^2=0,$$

which may be written

$$\begin{vmatrix} 2A, & 2H, & Ax^2+2Hx+B+2G\\ H, & B-(Ax^2+2Hx+B), & F\\ Ax^2+2Hx+B+2G, & 2F, & 2C\end{vmatrix}=0,$$

a cubic in $Ax^2+2Hx+B$, which reduces to a quadratic in the same quantity, if $\begin{vmatrix} AHG\\ HBF\\ GFC\end{vmatrix}=0$; that is, if the discriminant to the determinant of $au+a'v+a''w$, &c., vanishes. Hence, if this condition is satisfied for p, q, r, it will also be satisfied by any three homogeneous linear functions of p, q, r.

9090. (S. TEBAY, B.A.)—If a, b, c be conterminous edges of a tetrahedron, x, y, z the opposite edges; A, B, C, X, Y, Z the dihedral angles respectively opposite to these edges; A_1, A_2, A_3, A_4 the areas of the faces, and $M = (1-\cos^2 X-\cos^2 Y-\cos^2 Z-2\cos X\cos Y\cos Z)^{\frac{1}{2}}$;

prove that $V = \frac{1}{3}(2MA_1A_2A_3)^{\frac{1}{2}} = \frac{3}{2}A_4^2/(x\cot A + y\cot B + z\cot C)$.

Solution by Professor WOLSTENHOLME, M.A., Sc.D.

The tetrahedron being OABC, $OA = a$, $OB = b$, $OC = c$; $BC = x$, $CA = y$, $AB = z$; let fall OO' $(= p_1)$ perpendicular on ABC, $O'a$, $O'b$, $O'c$ perpendiculars from O' on BC, CA, AB; then

$$O'a = p_1\cot A,\quad O'b = p_1\cot B,\quad O'c = p_1\cot C;$$

also
$$A_1 = \tfrac{1}{2}(x\,.\,O'a + y\,.\,O'b + z\,.\,O'c),$$
$$= \tfrac{1}{2}p_1(x\cot A + y\cot B + z\cot C);$$

but $V = \frac{1}{3}A_1p_1$, whence we obtain

$$V = \tfrac{2}{3}A_1^2\,solidus\,(x\cot A + y\cot B + z\cot C).$$

Again, $V = \frac{1}{6}abc(1-\cos^2\alpha_1-\cos^2\beta_1-\cos^2\gamma_1+2\cos\alpha_1\cos\beta_1\cos\gamma_1)^{\frac{1}{2}}$,

when
$$\alpha_1 = BOC,\ \&c.,$$
$$A_2 = \tfrac{1}{2}bc\sin\alpha_1,\quad A_3 = \tfrac{1}{2}ca\sin\beta_1,\quad A_4 = \tfrac{1}{2}ab\sin\gamma_1,$$

or $V = \frac{1}{3}\sqrt{2}\,(A_2A_3A_4)^{\frac{1}{2}}$
$$\frac{(1-\cos^2\alpha_1-\cos^2\beta_1-\cos^2\gamma_1+2\cos\alpha_1\cos\beta_1\cos\gamma_1)^{\frac{1}{2}}}{(\sin\alpha_1\sin\beta_1\sin\gamma_1)^{\frac{1}{2}}}.$$

But, by ordinary spherical Trigonometry formulæ,

$$\sin\beta_1\sin\gamma_1\sin X = (1-\cos^2\alpha_1-\cos^2\beta_1-\cos^2\gamma_1+2\cos\alpha_1\cos\beta_1\cos\gamma_1)^{\frac{1}{2}}$$
$$(\equiv n_1) = \sin\gamma_1\sin\alpha_1\sin Y = \sin\alpha_1\sin\beta_1\sin Z;$$

and $\sin Y\sin Z\sin\alpha_1 = (1-\cos^2X-\cos^2Y-\cos^2Z-2\cos X\cos Y\cos Z)^{\frac{1}{2}}$
$$(\equiv N_1) = \sin Z\sin X\sin\beta_1 = \sin X\sin Y\sin\gamma_1;$$

whence we obtain

$$\sin\alpha_1\sin\beta_1\sin\gamma_1 = \frac{n_1^2}{N_1},\quad \text{or}\quad \frac{n_1}{(\sin\alpha_1\sin\beta_1\sin\gamma_1)^{\frac{1}{2}}} = \sqrt{N_1},$$

and, finally,
$$V = \tfrac{1}{3}(2N_1A_2A_3A_4)^{\frac{1}{2}}.$$

[Here A_1, A_2, A_3, A_4 are taken to be the areas of the faces opposite to O, A, B, C. Mr. TEBAY has taken these as A_4, A_1, A_2, A_3.]

Hence, we get $\dfrac{A_1}{N_1} = \dfrac{A_2}{N_2} = \dfrac{A_3}{N_3} = \dfrac{A_4}{N_4} = \dfrac{2A_1A_2A_3A_4}{9V^2}$,

The equalities $\dfrac{A_1}{N_1} = \dfrac{A_2}{N_2} = \ldots$ are, however, more easily obtained from the four equations, obtained by projecting on each of the four faces in turn,

$$A_1 = A_2\cos X + A_3\cos Y + A_4\cos Z \quad\ldots\ldots(1),$$
$$A_2 = A_1\cos X + A_3\cos C + A_4\cos B \quad\ldots\ldots(2),$$
$$A_3 = A_1\cos Y + A_2\cos C + A_4\cos A \quad\ldots\ldots(3),$$
$$A_4 = A_1\cos Z + A_2\cos B + A_3\cos A \quad\ldots\ldots(4).$$

If we eliminate A_3 and A_4 from (2), (3), (4); and also from (1), (3), (4), we get two equations for $A_1 : A_2$; which will be found to give $\frac{A_1^2}{N_1^2} = \frac{A_2^2}{N_2^2}$.]

9106. (W. J. Greenstreet, B.A.)—There are 64 competitors for the Yorkshire Challenge Cup (at Football). They are drawn in pairs, and the winners of first round are drawn in pairs, and so on. Prove that Hull's chance of playing Bradford (1) in the final is $\frac{1}{2016}$, and (2) in the course of the contest is $\frac{1}{32}$.

Solution by D. Biddle.

Assuming the chances of winning and losing to be always equal, the probabilities for the several rounds will be as follows:—

	That Hull plays Bradford.	That they do not meet, but both win.
I.	$\frac{1}{64}$	$\frac{63}{64} \cdot \frac{1}{4} = \frac{31}{128}$.
II.	$\frac{31}{128} \cdot \frac{1}{31} = \frac{1}{128}$	$\frac{31}{128} \cdot \frac{30}{31} \cdot \frac{1}{4} = \frac{15}{252}$.
III.	$\frac{15}{252} \cdot \frac{1}{15} = \frac{1}{252}$	$\frac{15}{252} \cdot \frac{14}{15} \cdot \frac{1}{4} = \frac{7}{504}$.
IV.	$\frac{7}{504} \cdot \frac{1}{7} = \frac{1}{504}$	$\frac{7}{504} \cdot \frac{6}{7} \cdot \frac{1}{4} = \frac{3}{1008}$.
V.	$\frac{3}{1008} \cdot \frac{1}{3} = \frac{1}{1008}$	$\frac{2}{1008} \cdot \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{2016}$.
VI.	$\frac{1}{2016} \cdot \frac{1}{1} = \frac{1}{2016}$.	

Hence the answer to (1) is $\frac{1}{2016}$, whilst that to (2) is

$$\frac{1}{64} + \frac{1}{128} + \frac{1}{252} + \frac{1}{504} + \frac{1}{1008} + \frac{1}{2016} = \frac{63}{2016} = \frac{1}{32}.$$

6987 & 9077. (Professor Steggall, M.A.)—(9077.) Show that there are twelve ways of placing eight queens on a chess-board, so that none are *en prise;* and that, of these, six are mutually derivable by uniform horizontal or vertical shifting, four are absolutely independent, and one is semi-symmetrical.

6987. (B. Reynolds, M.A.)—Find (1) in how many essentially different ways eight queens can be placed on an ordinary chess-board, so that no two can take each other; and show (2) that seven queens can be placed on a seven-square board in seven ways, so as not to be able to take each other.

Solution by B. Reynolds, M.A.; H. Fortey, M.A.; *and others.*

Let the notation 42857136 denote that queens are to be placed on QR4, QKt2, QB8, Q5, and so on. Then the six associated solutions may be represented thus:

36428571	47531682
64285713	75316824
42857136	53168247.

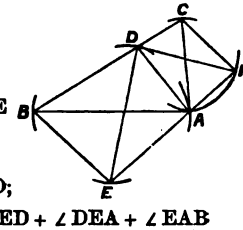
The semi-symmetrical solution thus: 46827135. The remaining five (making twelve in all) may be suggestively written thus:

518, 63, 724	168, 37, 425
518, 427, 36	425, 861, 37.
427, 518, 63	

[There are twelve primary positions, of which eleven are unsymmetrical and one semi-symmetrical. Each primary unsymmetrical position is four-fold as the board is turned round, and each of the four generates another position by converting a right-hand into a left-hand pattern. The primary semi-symmetrical position is duplex as the board is turned round, and each of the two generates another as above. Therefore the number of positions is $= 11 \times 8 + 1 \cdot 4 = 92$. Of the eleven unsymmetrical primary positions seven can be deduced from one another by uniform shifting. The other four and the semi-symmetrical position are independent. Under Quest. 6987 the problem has been completely solved on p. 71 of Vol. XXXVII. of our *Mathematics from the Educational Times;* it is also (as we have recently ascertained) worked out in LUCAS's *Récréations Mathématiques*, Vol. I., pp. 59—86, and in JAENISCH's *Application de l'Analyse Mathématique au Jeu des Echecs*, Vol. I., pp. 126—135.]

8753. (Professor HUDSON, M.A.)—ABC is a right-angled triangle; a point D is taken in the hypothenuse BC, and circles are described about ABD, ACD. If E, F be the mid-points of the arcs AB, AC remote from D, prove that EAF is a straight line.

Solution by W. J. GREENSTREET, B.A.; A. H. LEWES, M.A.; *and others.*



Since E is mid-point of arc BA, $\angle ADB = \angle EDA$, therefore it follows that $\angle BAE = \angle ABE$. Similarly $\angle FDA = \angle CDF = \angle CAF$ and $= ACF$. Now

$$\angle EAD = \angle BAD + \angle BAE = \angle BED + \angle BAE$$

$$\angle DAF = 180° - \angle ADF - \angle AFD$$
$$= 90° - \angle CAF + 90° - \angle ACD$$
$$= \angle BAE \text{ or } \angle ABE + \angle ABD \text{ or } \angle AED;$$

therefore $\angle EAD + \angle DAF = \angle ABE + \angle BED + \angle DEA + \angle EAB$
$= \text{two right angles,}$

therefore E, A, F are collinear.

9055. (Captain H. BROCARD.)—Le cercle de Brocard étant le lieu des centres des symédianes (point de Lemoine) des triangles circonscrits et semblables au triangle donné ABC, démontrer que l'angle ϕ des côtés du

triangle T_ϕ avec ceux du triangle ABC n'est autre que l'angle que fait avec OK la ligne qui joint O au centre K'_ϕ des symédianes du triangle T_ϕ.

Solution by Prof. NASH, M.A.; Prof. MATZ, M.A.; *and others.*

ABC is a Tucker triangle of the circumscribed triangle T_ϕ, ω is the centre of similitude, and the angle subtended at ω by corresponding points is ϕ. Hence $O\omega O_\phi = K\omega K_\phi = \phi$. But $OK_\phi\omega = OK\omega = \frac{1}{2}\pi - \omega$, therefore $O\omega K K_\phi$ are concyclic, therefore $KOK_\phi = K\omega K = \phi$.

8944. (W. J. C. SHARP, M.A.)—Form the equation to the two tangent planes, which can in general be drawn to a given quadric, so as to contain a given straight line.

Solution by Professor SEBASTIAN SIRCOM, M.A.

Let the equations of the line be

$$U \equiv \alpha x + \beta y + \gamma z + \delta = 0, \quad V \equiv \alpha_1 x + \beta_1 y + \gamma_1 z + \delta_1 = 0.$$

The condition that the plane $u + \lambda v = 0$ shall touch the given quadric is $U + \lambda P + U_1\lambda^2 = 0$, where

$$U \equiv A\alpha^2 + B\beta^2 + C\gamma^2 + D\delta^2 + 2F\beta\gamma + 2G\gamma\alpha + 2H\alpha\beta + 2L\alpha\delta + 2M\beta\delta + 2N\gamma\delta,$$

$$U_1 \equiv A\alpha_1^2 + \&c. \text{ and } P \equiv \alpha_1 \frac{dU}{d\alpha} + \beta_1 \frac{dU}{d\beta} + \gamma_1 \frac{dU}{d\gamma} + \delta_1 \frac{dU}{d\delta},$$

the coefficients A, &c., having their usual meanings.

Hence the equation of the tangent planes through the given line is $Uv^2 - Puv + U_1u^2 = 0$.

9043. (Rev. T. C. SIMMONS, M.A.)—In a bag is a particular coin whose value is M, also a number of other coins whose aggregate value is m. A person being allowed to draw one at a time till he draws the coin M, when the drawing is to cease, show that the value of his expectation is $M + \frac{1}{2}m$.

Solution by H. FORTEY, M.A.; SARAH MARKS, B.Sc.; *and others.*

Let *all* the coins be drawn one by one, but only M and those drawn before it be retained. This will not affect the drawer's expectation. Let the number of coins, including M, be $n+1$, and call the other coins $a_1, a_2, \ldots a_n$. Then, chance of drawing them in any order $a_1, a_2 \ldots a_r$, M, $a_{r+1} \ldots a_n$ is $\frac{1}{(n+1)!}$, and chance of drawing them in the reverse order is the same; and expectation from this pair of possible drawings

$$= \frac{1}{(n+1)!}(a_1 + a_2 + \ldots a_r + M + M + a_{r+1} + \ldots a_n) = \frac{1}{(n+1)!}(2M + m).$$

But there are $\frac{1}{2}(n+1)!$ such pairs of possible drawings, therefore expectation $= \frac{1}{2}(2M+m) = M+\frac{1}{2}m$.

[The expectation from first drawing $= \frac{M+m}{n+1}$. The chance of a second drawing $= \frac{n}{n+1}$, and aggregate value of coins now left $= M+\frac{n-1}{n}.m$, so that the expectation from the second drawing is

$$\frac{1}{n}.\frac{n}{n+1}\left\{M+\frac{n-1}{n}.m\right\}.$$

Similarly, the expectation from the third drawing is

$$\frac{1}{n-1}.\frac{(n-1)n}{n(n+1)}\left\{M+\frac{n-2}{n}.m\right\};$$

and thus the whole expectation is

$$\frac{M}{n+1}+\frac{M}{n+1}+\ldots \text{ to } n+1 \text{ terms } +\frac{m}{n+1}\left\{1+\frac{n-1}{n}+\frac{n-2}{n}+\ldots+\frac{1}{n}\right\}$$
$$= M+\tfrac{1}{2}m.]$$

9008. (Samuel Roberts, M.A.)—Given three circles C_1, C_2, C_3, determine a circle cutting C_1 orthogonally, bisecting C_2, and bisected by C_3; and show that in general there are two such circles which may coincide or become imaginary.

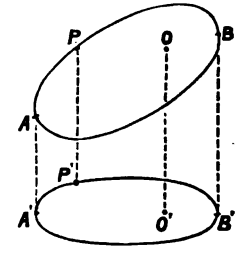
Solution by Alfred Harker, M.A.; W. J. C. Sharp, M.A.; *and others.*

If C be the centre of the required circle, r its radius, r_1, r_2, r_3 the radii of the given circles; then $CC_1^2 = r^2+r_1^2$; $CC_2^2 = r^2-r_2^2$; $CC_3^2 = r_3^2-r^2$. Hence $CC_1^2+CC_3^2 = r_1^2+r_3^2$; or, if M be the middle point of C_1C_3, $2CM^2+2C_1M^2 = r_1^2+r_3^2$, and C lies on a circle having M for centre. Again, $CC_2^2+CC_3^2 = r_3^2-r_2^2$; or, if N be the middle point of C_2C_3, C lies on another circle having N for centre. Since C is thus given as the intersection of two circles, there are two solutions, and the other results follow.

9070. (R. Curtis, M.A.)—A particle moving freely round a centre of force O is projected into a particle moving freely round a centre of force O'; show that (1) O' is the projection of O; (2) if both orbits be conics, v and F the velocity and force exerted on the particle in the first orbit, v' and F' in the second, α and β the semi-diameters parallel to v and F in first orbit, α' and β' those parallel to v' and F' in the second orbit, $v/\alpha = v'/\alpha'$, $F/\beta = F'/\beta'$; (3) hence prove Sir W. R. Hamilton's theorem, that the law of force for which a particle will describe an ellipse round any point S' in its plane as centre of force is $F = \frac{\mu.\beta^6}{SP^2.PQ^3}$, where P is the particle and PQ a chord drawn through S and terminated by the ellipse, and α a semi-diameter parallel to PQ.

Solution by F. X. De Wachter; A. Harker, M.A.; *and others.*

1. Let APB be the orbit described round O, and A′P′B′ the projected orbit.



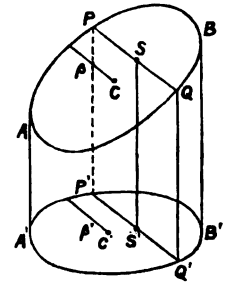
The plane passing through O and the moving generatrix BB′ contains invariably the axis OO′ along which O is projected into O′. Then, the areas described in both planes APB and A′P′B′ by the plane OO′BB′ revolving about OO′, will be proportional to the times, because the area described round O in APB bears a fixed ratio to that which is simultaneously described round O′ in A′P′B′, and the projecting particle, in virtue of its free motion, is supposed to describe equal areas in equal times round O.

Now, the same property applying to the projected particle moving round O′, it is impossible to derive its motion from any other centre of forces, but from O′ projection of O.

2. Of course v' is the projection of v. Besides, their respective directions are parallel to the semi-diameters α' and α, α' being similarly the projection of α. Therefore $v'/v = \alpha'/\alpha$ or $v/\alpha = v'/\alpha'$.

The forces F and F′ bear to one another the same ratio as the arrows (*sagittae*) corresponding to the arcs and areas that are simultaneously described in both orbits, when these arrows are taken *at the limit of evanescence.* (Newton's expression.) But, for any two arrows corresponding to any two arcs simultaneously described in APB and A′P′B′, it is obvious that the latter is the projection of the former. Now, if through the centres we draw the semi-diameters β and β' parallel to the directions of those arrows, that is to say, to the directions of the forces F and F′, we see directly that β' is the projection of β, and, consequently, that the forces F and F′ are proportional to β and β' or $F/\beta = F'/\beta'$.

3. Any ellipse APB may be projected into a circle, say A′P′B′.



Now, in a free circular motion round any point S′ within the orbit, we have, according to Newton (*Principia* I., Prop. 7),

$$F' = \frac{K}{S'P'^2 \cdot P'Q'^3}.$$

But we have also

$$F/F' = \beta/\beta' = SP/S'P' = PQ/P'Q'.$$

Hence we may eliminate F′, S′P′, and P′Q′ from the above formula, and express F by means of SP, PQ, and β, β' being a constant.

Thus we get $$F = \frac{\mu\beta^6}{SP^2 \cdot PQ^3}.$$

9065. (R. W. D. Christie, M.A.)—If $\Sigma^2 = 1^5 + 2^5 + 3^5 \ldots n^5$, show that the first solution in integers is $\Sigma = 1001$, and find another.

Solution by H. FORTEY, M.A.; Prof. BEYENS, M.A.; *and others.*

We have $\Sigma^2 = \frac{1}{4}n^2(n+1)^2 \cdot \frac{1}{3}(2n^2+2n-1)$;

hence, if $2n^2+2n-1 = 3x^2$, then $(2n+1)^2 = 3(2x^2+1) = 9y^2$,

(when y is an integer), therefore $3y^2-2x^2 = 1$.

Every odd convergent to $\sqrt{\frac{3}{2}}$ will give an integral solution of this equation; the numerator being x and the denominator being y,

$x = 1, 11, 109, 1079, 10681$, &c., $y = 1, 9, 89, 881, 8721$, &c.

(if x_p be the p^{th} value of x, $x_p = 10x_{p-1}-x_{p-2}$, and the values of y follow the same law). Also $n = \frac{1}{2}(3y-1)$, $\Sigma = \frac{1}{8}x(9y^2-1)$, and corresponding values of n and Σ are

$n = 1, 13, 133, 1321, 13081$, &c.

$\Sigma = 1, 1001, 971299, 942162299, 913896491101$, &c.

8642. (Professor MATHEWS, M.A.)—Give an elementary geometrical method for constructing the spheres which pass through two given points, and touch two given planes.

Solution by the PROPOSER.

Construction of spheres through two given points A, B, and touching two given planes α, β.

Let the plane bisecting AB at right angles meet one of the bisectors of the dihedral angle $\alpha\beta$ in a line l. Then the spheres through A, B and having their centres on l will form a coaxal system. The section of this by the plane α will be a coaxal system of circles. Let P, Q be the point-circles of this system. Construct the spheres with centres on l, and passing through (P, A, B), (Q, A, B) respectively. These will satisfy the required conditions, and, since there are two bisectors of the dihedral angle $\alpha\beta$, we shall have four spheres altogether.

9063. (MAHENDRA NATH RAY, M.A., LL.B.)—If $a_1, a_2, a_3 \ldots a_{2n-1}$ be $2n-1$ positive numbers connected by the relation $a_1a_2a_3 \ldots a_{2n-1} = 1$; show, *by elementary algebra only*, that the minimum value of

$$(1+a_1)(1+a_2)(1+a_3)(1+a_4) \ldots (1+a_{2n-1}) \text{ is } 2^{2n-1}.$$

Solution by MUNINDRA BHATTACHARYYA; J. M. STOOPS, M.A.; *and others.*

Since $(1+a_1)^2$ is never less than $4a_1$, $(1+a_1)/\sqrt{a_1}$ is never less than 2, and its minimum value is 2; similarly, the minimum value of $(1+a_2)/\sqrt{a_2}$ is 2; and so on. Hence the minimum value of

$$\{(1+a_1)(1+a_2) \ldots (1+a_n)\} / (a_1a_2 \ldots a_n)^{\frac{1}{2}} \text{ is } 2^n.$$

And, the denominator being by hypothesis $= 1$, the minimum value of

$$(1+a_1)(1+a_2) \ldots (1+a_n) \text{ is } 2^n.$$

[The restricted form $(2n-1)$ given in the question is unnecessary.]

8569. (J. O'Byrne Croke, M.A.)—Prove that the square of the tangent of half the vertical angle of an isosceles triangle in terms of the lengths of the segments s_1, s_2; σ_1, σ_2, of any two straight lines drawn through the middle point of the base and terminated by the sides, is equal to $\left\{\left(\frac{1}{s_1}-\frac{1}{s_2}\right)^2-\left(\frac{1}{\sigma_1}-\frac{1}{\sigma_2}\right)^2\right\}\Big/\left\{\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}\right)^2-\left(\frac{1}{s_1}+\frac{1}{s_2}\right)^2\right\}$.

Solution by the Proposer; Professor Ignacio Beyens; *and others.*

Let h = height, b = semi-base, and α = semi-vertical angle of the triangle; and let ϕ = angle between base and the line whose segments are s_1, s_2; then

$$\frac{s_1}{h}=\frac{\sin\alpha}{\cos(\alpha-\phi)},\quad \frac{s_2}{h}=\frac{\sin\alpha}{\cos(\alpha+\phi)},\quad \frac{1}{s_1}=\frac{1}{h\tan\alpha}(\cos\phi+\sin\phi\tan\alpha),$$

$$\frac{1}{s_2}=\frac{1}{h\tan\alpha}(\cos\phi-\sin\phi\tan\alpha),$$

therefore $$\frac{1}{s_1}+\frac{1}{s_2}=\frac{2\cos\phi}{b},\quad \frac{1}{s_1}-\frac{1}{s_2}=\frac{2\sin\phi}{h},$$

therefore $$b^2\left(\frac{1}{s_1}+\frac{1}{s_2}\right)^2+h^2\left(\frac{1}{s_1}-\frac{1}{s_2}\right)^2=4 \quad\ldots\ldots\ldots\ldots(1),$$

the general relation between segments of line through mid-base, semi-base, and perpendicular. Also,

$$b^2\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}\right)^2+h^2\left(\frac{1}{\sigma_1}-\frac{1}{\sigma_2}\right)^2=4 \quad\ldots\ldots\ldots\ldots(2).$$

Hence, by subtraction of (1) and (2), transposition, and division, we have the required result.

7902. (R. Rawson, F.R.A.S., &c.)—Show that the general integral of

$$A_2^{\frac{1}{2}}\left(1+B_2x^2+\frac{B_2^2A_4}{A_2^2}x^4\right)^{\frac{1}{2}}.\frac{dy}{dx}+B_2^{\frac{1}{2}}(1+A_2y^2+A_4y^4)^{\frac{1}{2}}=0$$

is $[CA_2+B_2A_4x^2]\,y^2+2\,[(A_2B_2)^{\frac{1}{2}}(A_4+C^2-CA_2)^{\frac{1}{2}}]\,xy+A_2+CB_2x^2=0.$

Solutions by (1) D. Edwardes; (2) *the* Proposer.

1. Write the equation

$$\frac{B_2^{\frac{1}{2}}dx}{X^{\frac{1}{2}}}+\frac{A_2^{\frac{1}{2}}dy}{Y^{\frac{1}{2}}}=0,\quad \text{say,}\quad Mdx+Ndy=0.$$

Let $P=A_2-B_2A_4x^2y^2$, then $\dfrac{B_2^{\frac{1}{2}}Y^{\frac{1}{2}}dx}{P}+\dfrac{A_2^{\frac{1}{2}}X^{\frac{1}{2}}dy}{P}=0.$

Integrating by parts, we have

$$\frac{B_2^{\frac{1}{2}}Y^{\frac{1}{2}}x+A_2^{\frac{1}{2}}X^{\frac{1}{2}}y}{P}=\text{constant}+\int\left\{B_2^{\frac{1}{2}}x\,d\left(\frac{Y^{\frac{1}{2}}}{P}\right)+A_2^{\frac{1}{2}}y\,d\left(\frac{X^{\frac{1}{2}}}{P}\right)\right\}.$$

Performing the differentiation, the part under the integral sign becomes, after some obvious reductions,

$$(M\,dx + N\,dy)\left\{\frac{(A_2B_2)^{\frac{1}{2}}\,xy}{P} + \frac{2B_2A_2x^2y^2(XY)^{\frac{1}{2}}}{P^2} + \left(\frac{B_2}{A_2}\right)^{\frac{1}{2}}\frac{B_2x^2 + A_2y^2 + A_2B_2x^2y^2}{P^2}\right\},$$

and therefore vanishes, and we are left with

$$\frac{B_2^{\frac{1}{2}}Y^{\frac{1}{2}}x + A_2^{\frac{1}{2}}X^{\frac{1}{2}}y}{P} = \text{constant},$$

which is a form of the general integral. Let, now, C be an arbitrary constant, and write $\dfrac{B_2^{\frac{1}{2}}xY^{\frac{1}{2}} + A_2^{\frac{1}{2}}X^{\frac{1}{2}}y}{P} = \left(-\dfrac{1}{CA_2}\right)^{\frac{1}{2}}$.

Now, rationalizing, the factor P^2 divides out, and the rest can be written $[(CA_2 + B_2A_4x^2)\,y^2 + A_2 + CB_2x^2]^2 = 4A_2B_2\,(A_4 + C^2 - CA_2)\,x^2y^2$, therefore, &c. [A solution, by verification only, is given on p. 82 of Vol. xlii.]

2. The integrating factor 1 : P is usually found so as to satisfy the condition of integrability, but in this case Mr. D. Edwardes has ingeniously applied the integration by parts, and has found 1 : P so as to make the quantity under the sign of integration to vanish.

Apply this method to the general equation

$$\mu M\,dx + \mu N\,dy = 0 \quad \text{.....(1)},$$

then

$$\mu\,(Mx + Ny) = C\int\left\{\left(\mu x\frac{dM}{M\,dx} + x\frac{d\mu}{dx}\right) \times \left(M\,dx + \frac{\mu y\,dN/N\,dy + y\,d\mu/dy}{\mu x\,dM/M\,dx + x\,d\mu/dx}\cdot N\,dy\right)\right\}.$$

The quantity under the sign $\int$ vanishes when

$$x\frac{d\mu}{dx} - y\frac{d\mu}{dy} = \mu\left\{y\frac{dN}{N\,dy} - x\frac{dM}{M\,dx}\right\} \quad \text{.....(2)};$$

and the general integral of (1) becomes

$$\mu\,(Mx + Ny) = C \quad \text{.....(3)}.$$

Generally (2) is as difficult to solve as is the original equation (1), but in particular cases it can be solved.

If M is a function of y only, and N is a function of x only, then (2) becomes

$$x\frac{d\mu}{dx} - y\frac{d\mu}{dy} = 0 \quad \text{.....(4)},$$

the general solution of which is

$$\mu = \phi\,(xy) \quad \text{.....(5)}.$$

Hence it appears, from the above method, that the form of the general Eulerian integral is

$$\phi\,(xy)\,(Mx + Ny) = C \quad \text{.....(6)}.$$

9042. (H. L. Orchard, B.Sc., M.A.)—Prove that $1^3+2^3+3^3+\ldots+x^3$ is a factor of the expression $3x^8+12x^7+14x^6-7x^4+2x^2$.

Solution by D. E. Shorto, M.A.; R. Knowles, B.A.; *and others.*

$$\Sigma^x x^3 = \left\{\tfrac{1}{2}x(x+1)\right\}^2,$$

and $3x^8+12x^7+14x^6-7x^4+2x^2 = x^2(x+1)^2(3x^4+6x^3-x^2-4x+2)$.

Hence Σx^3 is a factor of the given expression.

8841. (L. J. Rogers, B.A.)—Show that the complete primitive of the reciprocant equation

$$9\left(\frac{dy}{dx}\right)^2\cdot\frac{d^4y}{dx^4}-45\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}\cdot\frac{d^3y}{dx^3}+40\left(\frac{d^2y}{dx^2}\right)^3=0$$

is

$$(x^2+2px+q)(y^2+2ry+s) = (p^2-q)(r^2-s).$$

Solution by the Proposer; Professor Matz, M.A.; *and others.*

Using the ordinary notation, t, a, b, c, &c., for dy/dx, d^2y/dx^2, &c., we know that the solution of $9a^2d-45abc+40b^3=0$ is

$$Ax^2+By^2+C+2Fy+2Gx+2Hxy=0.$$

Differentiating, and eliminating y, we get

$$(C'x^2-2G'x+A')(Bt^2+2Ht+A)=\Delta,$$

after some reductions; A′, B′, &c. being the coefficients in the tangential equation, and Δ the discriminant of the conic. Since $A'C'-G'^2=B\Delta$, we can easily reduce to the given form.

8811. (A. Russell, B.A.)—If the temperature of an infinite cube of isotropic matter be initially zero, and the faces be subsequently kept at temperatures $A_1\sin\frac{\pi y}{2l}\sinh\frac{\pi z}{2l}f_1(t)$, $A_2\sin\frac{\pi z}{2l}\sinh\frac{\pi x}{2l}f_2(t)$, &c., the axes being three edges of the cube; find the temperature at any subsequent period.

Solution by the Proposer; Professor Sarkar, M.A.; *and others.*

The solution is

$$v=\frac{2A_1}{\sqrt{\pi}}\sin\frac{\pi y}{2l}\sinh\frac{\pi z}{2l}\int_{x/(2a\sqrt{t})}^{\infty}f_1\left(t-\frac{x^2}{4a^2\gamma^2}\right)e^{-\gamma^2}d\gamma+\ldots+\ldots.$$

To verify this, it is only necessary to observe that this value of v satisfies equation $\frac{dv}{dt} = a^2\nabla^2 v$, and that (i.) when $t = 0$, $v = 0$, (ii.) $x = 0$, $v = A_1 \sin\frac{\pi y}{2l} \sinh\frac{\pi z}{2l} f_1(t)$, &c.

9021. (R. W. D. Christie, M.A.)—Prove that

$$\frac{\tan 52\frac{1}{2}^\circ + \tan 7\frac{1}{2}^\circ}{\tan 82\frac{1}{2}^\circ + \tan 37\frac{1}{2}^\circ} = \frac{\tan 52\frac{1}{2}^\circ - \tan 7\frac{1}{2}^\circ}{\tan 82\frac{1}{2}^\circ - \tan 37\frac{1}{2}^\circ} = \frac{\tan 22\frac{1}{2}^\circ}{\tan 67\frac{1}{2}^\circ} = (\sqrt{2}-1)^2.$$

Solution by Hannah Moylan; G. G. Storr, M.A.; *and others.*

$$\tan 7\tfrac{1}{2}^\circ = \sqrt{6}-\sqrt{3}+\sqrt{2}-2 = (\sqrt{3}-\sqrt{2})(\sqrt{2}-1),$$
$$\tan 37\tfrac{1}{2}^\circ = \sqrt{6}+\sqrt{3}-(\sqrt{2}+2) = (\sqrt{3}-\sqrt{2})(\sqrt{2}+1),$$
$$\tan 52\tfrac{1}{2}^\circ = \sqrt{6}-\sqrt{3}-(\sqrt{2}-2) = (\sqrt{3}+\sqrt{2})(\sqrt{2}-1),$$
$$\tan 82\tfrac{1}{2}^\circ = \sqrt{6}+\sqrt{3}+\sqrt{2}+2 = (\sqrt{3}+\sqrt{2})(\sqrt{2}+1).$$

The first two tangents are well known, and the other two are easily obtained from the equation $\tan x \tan y = 1$, where $x + y = 90°$. The result stated in the Question easily follows.

9064. (A. E. Thomas, M.A.)—If A, B, C are the angles of an acute-angled triangle, prove that $\tan^4 A + \tan^4 B + \tan^4 C \not< 27$,

$$\tan^4 A \tan^4 B + \tan^4 B \tan^4 C + \tan^4 C \tan^4 A \not< 243.$$

Solution by H. L. Orchard, B.Sc., M.A.; Gertrude Poole, B.A.; *and others.*

It will be found that, A, B, C, being acute,

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

is *least* when $A = B = C = 60°$. But *then* we have

$$\tan^4 A + \tan^4 B + \tan^4 C = 27,$$

and $$\tan^4 A \tan^4 B + \tan^4 B \tan^4 C + \tan^4 C \tan^4 A = 243,$$

whence the required results.

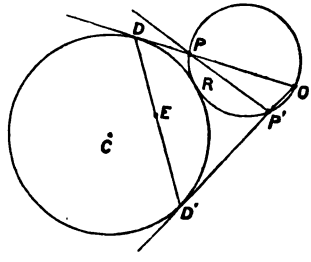
8848. (Asparagus.)—A circle is drawn through a fixed point O, touching a fixed circle internally or externally, and meeting in P, P′ the two tangents drawn from O to the fixed circle; prove that the envelope

of PP′ is a circle touching the two tangents drawn from O to the fixed circle and having its centre at the middle point of their chord of contact.

Solution by R. F. Davis, M.A.

Let R be the point of contact. Then, since the two circles together with a point-circle at R constitute a coaxal system,

$$OD : OR = PD : PR = P'D' : P'R$$
$$= \lambda : 1 \text{ (say)}.$$



Then the sum of the perpendiculars from D and D′ on PP′ is

$$= PD \sin OPP' + P'D' \sin OP'P$$
$$= \lambda (PR \sin OPP' + P'R \sin OP'P)$$
$$= \lambda OR \sin POP' = OD \sin POP'$$
$$= \text{constant}.$$

Hence PP′ touches a fixed circle having its centre at E, the middle point of DD′, and touching OD, OD′.

The above property also follows very easily by inversion from a theorem due to M. Mannheim:—"Si à un cercle I, inscrit à un angle aOb, on mène une tangente extérieure quelconque ab, la circonférence circonscrite au triangle Oab est tangente à une circonférence fixe de, inscrite à l'angle aOb" (Catalan).

8831. (Professor Hudson, M.A.)—Prove that the perpendicular from the orthocentre on the bisector of an angle of a triangle meets it on that diameter of the nine-points circle that passes through the mid-point of the opposite side.

Solution by C. E. Williams, M.A.; D. O. S. Davies; *and others.*

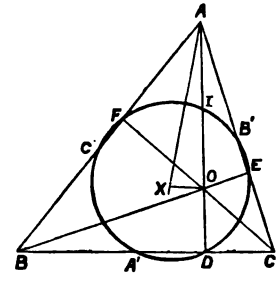
Let X be the foot of the perpendicular from O on the bisector AX. A circle passes round AFxOE of which I is the centre, and of which FX and XE are equal arcs, because the angle

$$FAX = EAX;$$

therefore $FX = EX$,

and $FI = EI$,

and therefore IX is a diameter of the nine-point circle, and it passes through A′ because IDA′ is a right angle.



7889. (Professor SYLVESTER, F.R.S.)—If by the developant of ω^2 matrices of order ω be understood in general the determinant to the square matrix formed by *developing* each of them into a line of ω^2 terms, and under- or over-writing these lines in regular order to one another, and if by the quantic associated to i such matrices, say $m_1, m_2, \ldots m_i$ be understood the quantic of order ω, in $x_1, x_2, \ldots x_i$, which is the determinant to the matrix $x_1m_1 + x_2m_2 + \ldots + x_im_i$; prove that the invariant of the quantic associated to four matrices of the second order is (at all events to a numerical multiplier *près*) the square of their developant.

Solution by Professor SEBASTIAN SIRCOM, M.A.

Let the matrices be $\begin{matrix} a_1, & b_1, & a_2, & b_2, & a_3, & b_3, & a_4, & b_4, \\ c_1, & d_1, & c_2, & d_2, & c_3, & d_3, & c_4, & d_4. \end{matrix}$

Their developant, D, is $\begin{vmatrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \\ c_1, & c_2, & c_3, & c_4 \\ d_1, & d_2, & d_3, & d_4 \end{vmatrix}$ and the associated quantic $\begin{vmatrix} a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4, & b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 \\ c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4, & d_1x_1 + d_2x_2 + d_3x_3 + d_4x_4 \end{vmatrix}$,

the discriminant of which, to a numerical factor *près*, is

$$\begin{vmatrix} 2a_1d_1 - 2b_1c_1, & a_1d_2 + a_2d_1 - b_1c_2 - b_2c_1 \ldots \\ \text{\&c.} \qquad\qquad \text{\&c.} \end{vmatrix} = \begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix} \begin{vmatrix} d_1, & -c_1, & -b_1, & a_1 \\ d_2, & -c_2, & -b_2, & a_2 \\ d_3, & -c_3, & -b_3, & a_3 \\ d_4, & -c_4, & -b_4, & a_4 \end{vmatrix} = D^2.$$

9093 & 9170. (S. ROBERTS, F.R.S.)—(9170.) Show that, (1) if the sides of a triangle (produced if necessary) are cut by a circle, and on the linear intercepts as diameters three circles are drawn, the radical centre of these is the "isogonal conjugate," relative to the triangle, of the centre of the first named circle; (2) if the planes of the faces of a tetrahedron are cut by a sphere, and on the circular intercepts as diametral sections four spheres are constructed, the radical centre of these spheres is the "isogonal conjugate" relative to the tetrahedron of the centre of the first named sphere.

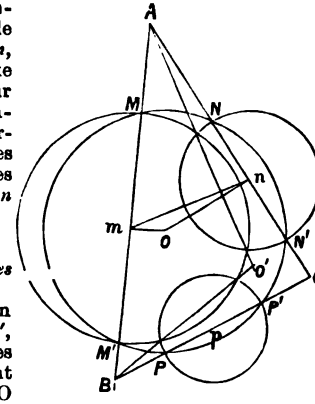
(9093.) Four spheres are bisected by the faces of a tetrahedron, and each of them passes through the three vertices on the corresponding face. Show that the sphere cutting these four spheres orthogonally has for its centre the tetrahedral inverse of the centre of the circumscribing sphere (*i.e.*, if $\alpha, \beta, \gamma, \delta$ are the tetrahedral coordinates of the circumscribing sphere, $1/\alpha, 1/\beta, 1/\gamma, 1/\delta$ are those of the orthogonal sphere), and that the sphere through the centres of the four spheres first mentioned has its centre at the middle point between those of the orthogonal and circumscribing spheres, and passes through the common section of those spheres. Compare the result with an analogous property of the nine-point circle in plane space.

Solution by Professor NEUBERG.

9170. (1) Supposons qu'une circonférence (O) coupe les côtés d'un angle BAC en M et M′, N et N′, et soient m, n les milieux de MM′, NN′. L'axe radical des circonférences qui ont pour diamètres MM′, NN′, est la perpendiculaire AO′ abaissée du point d'intersection des axes radicaux des cercles (O) et (m), (O) et (n) sur la ligne des centres mn. Mais le quadrilatère $AmOn$ étant inscriptible,

$$\angle NAO' = Onm = OAm;$$

donc les droites AO, AO′ sont *isogonales* par rapport à l'angle BAC



Si la circonférence (O) coupe BC en P et P′ et que p est le milieu de PP′, les axes radicaux des circonférences (m), (n), (p), prises deux à deux, sont les isogonales des droites AO, BO, CO par rapport aux trois angles A, B, C. Donc O et O′ sont conjugués isogonaux par rapport au triangle ABC.

(2) Considérons maintenant une sphère (O) et un angle dièdre δ. Un plan P mené par le centre O perpendiculairement à l'arête du dièdre, passe par les centres m, n des cercles suivant lesquels les faces du dièdre rencontrent la sphère O. Soient MM′, NN′ les diamètres de ces cercles situés dans le plan P. Le plan radical des sphères (m), (n), qui ont pour diamètres MM′, NN′, passe par l'arête de δ et est perpendiculaire à la droite mn ; donc, d'après ce qui a été démontré ci-dessus, ce plan radical et le plan mené par le centre de la sphère (O) et l'arête de δ sont *isogonaux* par rapport à l'angle dièdre δ.

Supposons maintenant qu'une sphère (O) coupe les faces d'un tétraèdre ABCD suivant quatre cercles ayant pour centres les projections O_a, O_b, O_c, O_d de O sur les faces, et prenons ces cercles pour sections diamétrales de quatre sphères (O_a), (O_b), (O_c), (O_d). Les plans radicaux de ces sphères prises à deux, et les plans menés par O et les arêtes du tétraèdre ABCD, sont isogonaux par rapport aux dièdres correspondants de ce tétraèdre. Donc le centre radical O′ des sphères (O_a) ... est le conjugué isogonal de O.

Remarque I.—Soient O'_a, O'_b, O'_c, O'_d les projections de O′ sur les faces du tétraèdre ABCD, et soient O″, O''_a... les milieux de OO′, $O_aO'_a$... Les huit points O_a et O'_a ..., projections des deux points inverses O, O′ sur les faces de ABCD, sont situés sur une même sphère (O″) dont le centre est au milieu de OO′.

Si M est un point commun aux deux sphères (O) et (O′), on a :

$$(MO)^2 + (MO')^2 = 2\,(MO'')^2 + 2\,(OO'')^2.$$

Or, si r est le rayon de la section du plan BCD par la sphère (O), on a

$$(MO)^2 = (OO_a)^2 + r^2, \quad (MO')^2 = (O'O_a)^2 - r^2;$$

donc

$$(O_aO)^2 + (O_aO')^2 = 2\,(MO'')^2 + 2\,(OO'')^2,$$

ce qui démontre que $MO'' = O_aO''$, et que les trois sphères (O), (O′), (O″) ont même plan radical.

Remarque II.—La sphère (O) peut couper le plan BCD suivant un cercle imaginaire ayant pour centre la projection O_a de O. Dans ce cas, on peut remplacer la sphère (O_a) par celle qui a pour centre O_a et coupe la sphère (O) orthogonalement, à la condition que la sphère (O'), au lieu d'être orthogonale à la sphère (O_a), coupe celle-ci suivant une section diamétrale.

Il faudra un changement analogue, si le rayon de la sphère (O') était de la forme $R\sqrt{-1}$.

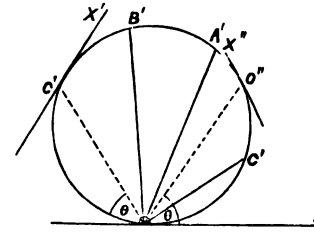
9093. Celle-ci est complétement résolue par la question 9170. Le point O' est, à certains égards, l'analogue de l'orthocentre du triangle, et la sphère (O'') est l'analogue du cercle des neuf points.

Pour ajouter une nouvelle analogie, nous dirons que O' est le centre de la sphère inscrite au tétraèdre $O'_aO'_bO'_cO'_d$.

9114. (Professor Neuberg.)—Etant donné un triangle quelconque ABC, on mène par un même point O trois droites OA', OB', OC' égales à BC, CA, AB et faisant avec un axe OX les angles XOA' = A, XOB' = B, XOC' = C. Démontrer: (1) que les points O, A', B', C' sont sur une circonférence égale à la circonférence ABC; (2) que les côtés du triangle A'B'C' sont doubles des distances entre les milieux des côtés de ABC et les pieds des hauteurs correspondantes; (3) que la droite joignant O au centre de gravité de ABC fait avec OX un angle complémentaire de l'angle de Brocard de ABC; (4) étant donné le triangle A'B'C', il existe trois triangles correspondants ABC et les trois points correspondants O sont les sommets d'un triangle équilatéral.

Solution by Samuel Roberts, F.R.S.

(1) Draw a circle equal to that circumscribing the triangle ABC and touching the axis OX at O. Let the lines OA', OB', OC' make with OX the angles XOA' = A, XOB' = B, XOC' = C. If A', B', C' are the points where these lines meet the circle touching OX at C, then we have OA' = BC, OB' = CA, OC' = AB. And conversely, if this is so, A', B', C', O must lie on a circle.



(2) The segment between the middle point of BC and the corresponding perpendicular has the value $\frac{1}{2}(AC\cos C - AB\cos B)$. But the value of B'C' is $AC\cos C + AB\cos(\pi - B)$. Therefore, &c.

(3) If G is the centre of gravity of A'B'C', we have, immediately,

$$\tan GOX = \frac{BC\sin A + CA\sin B + AB\sin C}{BC\cos A + CA\cos B + AB\cos C},$$

but this is one of various forms of the expression for the cotangent of the Brocard-angle for ABC, and implies that that angle and GOX are complementary.

(4) Suppose that O'O makes the angle θ with OX, and draw the tangent to the circle at O' forming an axis OX'. Then

$$X'O'A' = \pi - \theta - A, \quad X'O'B' = \pi - \theta - B, \quad X'O'C' = \pi - \theta - C,$$

and the sum of these angles is π if $\theta = \frac{1}{2}\pi$. But we may also take the point O'', so that

$$X''O''B' = B - \theta, \quad X''O''A' = A - \theta, \quad X''O''C' = \pi - (\theta - C),$$

and the sum of the angles is π, if $\theta = \frac{1}{2}\pi$. The two points so taken and O form therefore an equilateral triangle.

9078. (Professor Ignacio Beyens.)—Si les axes des coordonnées OX, OY, OZ sont tels que $XOY = XOZ = YOZ = 60°$, et si $yz + xy + xz + a^2 = 0$ est l'équation d'une surface rapportée à ces axes, et $x + y + z = 0$ un plan dans le même système des coordonnées, l'intersection du plan et de la surface est un cercle de rayon a.

Solution by Professor G. de Longchamps.

On observe que l'on a

$$(A) \quad yz + xy + xz + a^2 \equiv (x + y + z)^2 - (x^2 + y^2 + z^2 + yz + xy + xz - a^2).$$

D'ailleurs les angles des axes de coordonnées étant égaux à 60°, il en résulte que $x^2 + y^2 + z^2 + yz + xy + xz - a^2 = 0$ représente, dans ce système oblique, une sphère de rayon a ayant son centre à l'origine des coordonnées. La section de la quadrique par le plan $x + y + z = 0$ est donc, d'après (A), un cercle de rayon a.

8756. (Professor Wolstenholme, M.A., Sc.D.)—PSQ is a focal chord of a conic, and from S', the other focus, perpendiculars are drawn upon the normals at P, Q; prove that the straight line joining the feet of these perpendiculars will be parallel to PQ, and will pass through the centre of the conic.

Solution by R. Knowles, B.A.; Sarah Marks, B.Sc.; *and others.*

Let C be the centre of the conic; F, F' the feet of the perpendiculars; (x_1, y_1), (x_2, y_2) the coordinates of P and Q; (h, k) those of the pole of PQ. The equations to the normal at P and S'F are, respectively,

$$a^2y_1x - b^2x_1y = a^2e^2x_1y_1, \quad b^2x_1x + a^2y_1y = -ab^2ex_1;$$

whence the coordinates of F are

$$ex_1(ae - x_1)/(a - ex_1), \quad -ex_1y_1/(a - ex_1),$$

and similarly for F'. The equations to CF, CF' are, respectively,

$$(ae - x_1)y = -y_1x, \quad (ae - x_2)y = -y_2x \quad \ldots\ldots\ldots\ldots\ldots\ldots (1, 2).$$

In order that these may represent the same line, we must have

$$y_1(ae - x_2) = y_2(ae - x_1), \text{ or } ae(y_1 - y_2) = y_1x_2 - x_1y_2 = \frac{a^2}{h}(y_1 - y_2);$$

therefore these are identical when $h = a/e$, or PQ a focal chord, therefore FF′ passes through the centre. The equation to SP is

$$y = \frac{y_1}{x_1 - ae}(x - ae);$$

therefore CFF′ is parallel to PSQ.

8704. (A. Gordon.)—If a rod is hung horizontally by two parallel strings at its extremities, and displaced slightly, (1) along the direction of the rod, (2) by a small rotation round the vertical through its centre, prove that the ratio of the times of oscillation is constant and $= 3^{\frac{1}{2}} : 1$.

Solution by Rev. T. R. Terry, M.A.; Professor Matz, M.A.; *and others.*

Let M be the mass and $2l$ the length of the rod, and a the length of each string. In first case, if strings be displaced through small angle θ,

Kinetic energy lost $= \frac{1}{2}Ma^2\theta'^2$, Potential energy gained $= \frac{1}{2}Mav^2g$;

whence $a\theta'' = g\theta$ or Time $= 2\pi(a/g)^{\frac{1}{2}}$.

In second case, if rod be displaced through small angle ϕ,

Kinetic energy lost $= \frac{1}{2}M(\frac{1}{3}l)\phi'^2$,

Potential energy gained $= Mg\{a - (a^2 - l^2\phi^2)^{\frac{1}{2}}\} = \frac{1}{2}Mg(l^2/a)\phi^2$;

whence $a\phi'' = 3g\phi$, or Time $= 2\pi(a/3g)^{\frac{1}{2}}$.

9100. (W. J. Barton, M.A.)—Prove that the series

$$1 + \frac{1}{n}\cdot\frac{1}{3} + \frac{3n+1}{n(n+1)}\cdot\frac{1}{3^2} + \frac{(3n+1)(3n+4)}{n(n+1)(n+2)}\cdot\frac{1}{3^3} + \ldots\ldots$$
$$+ \frac{(3n+1)(3n+4)(3n+7)\ldots(6n-5)}{n(n+1)(n+2)\ldots(2n-1)}\cdot\frac{1}{3^n}$$
$$= \frac{(3n+1)(3n+4)\ldots(6n-2)}{n(n+1)\ldots(2n-1)}\cdot\frac{1}{3^n}.$$

Solution by Prof. Sircom, M.A.; H. Fortey, M.A.; *and others.*

The last term may be written

$$\frac{(3n+1)(3n+4)\ldots(6n-5)(6n-2)}{3n\,.\,3(n+1)\ldots 6n-3} - \frac{(3n+1)(3n+4)\ldots(6n-5)}{3n\,.\,3(n+1)\ldots 6n-6},$$

and so for the other terms; the sum is therefore as stated.

9066. (Rev. T. R. Terry, M.A.)—Prove the following property of the Gamma Function :—$\Gamma(\frac{1}{2}m)\,\Gamma\frac{1}{2}(m+1) = 2^{1-m}\,\Gamma(\frac{1}{2})\,\Gamma(m)$.

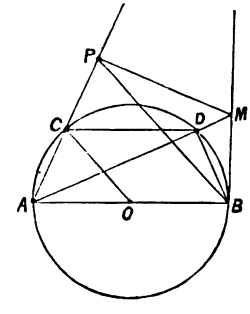
Solution by Isabel Maddison ; D. O. S. Davies, M.A. ; *and others.*

$$\Gamma(\tfrac{1}{2}m)\,\Gamma\tfrac{1}{2}(m+1) = \frac{1}{2^{m-1}}(m-1)\,(m-2)\ldots 1\,.\,\Gamma(\tfrac{1}{2}) = 2^{1-m}\,\Gamma(m)\,\Gamma(\tfrac{1}{2}).$$

[This solution assumes that m is a positive integer, whereas the theorem is true when m is any positive quantity. In the cases where the formula is practically useful, m is generally a positive proper fraction.]

8917. (Maurice D'Ocagne.)—On prend dans un cercle un diamètre AB et une corde CD parallèle à AB. La droite AD coupe en M la tangente menée au cercle par le point B. Démontrer que, si P est le pied de la perpendiculaire abaissée du point M sur la droite AC, (1) on a AP = 2AC ; et (2) lorsque la corde CD varie dans le cercle en restant paral èle au diamètre AB le lieu du point P est le cercle tangent en A au cercle donné et de diamètre double de celui-ci.

Solution by R. F. Davis, M.A. ; D. O. S. Davies ; *and others.*



Let O be the centre; join OC, BD, BP. Since A, P, M, B are concyclic,

$$\angle APB = \angle AMB = \angle ABD = \angle OAC = \angle OCA,$$

and BP is therefore parallel to OC.

But $AB = 2AO$;

hence (1) $AP = 2AC$,

and (2) $BP = 2OC = AB$,

and the locus of P is a circle centre B touching the given circle in A.

8984. (Professor Hudson, M.A.) — Prove that the locus of the feet of the perpendiculars from the origin on the chords of $xy = a^2$ (rectangular axes) that subtend half a right angle at the origin is

$$r^4 - 2a^2r^2\sin 2\theta = a^4.$$

Solution by R. Knowles, B.A. ; C. E. Williams, M.A. ; *and others.*

Let $x\cos\theta + y\sin\theta = r$ be such a chord. Two straight lines through the origin and its intersection with the curve are $r^2xy = (x\cos\theta + y\sin\theta)^2a^2$,

and if these be $y = m_1 x$, $y = m_2 x$, we have $m_1 - m_2 = 1 + m_1 m_2$, when $m_1 m_2 = \cot^2 \theta$, $m_1 + m_2 = (r^2 - 2a^2 \sin\theta \cos\theta) / (a^2 \sin^2 \theta)$. Eliminating m_1, m_2 between these three equations, we get the result.

9054. (J. Brill, M.A.)—A particle of mass m moves in a conservative field of force subject to the action of a retarding force varying as the velocity ($k \times$ velocity); prove that, if D denote the quantity of energy dissipated during a time t, $\frac{dD}{dt} + \frac{2k}{m}(D + V - E) = 0$;

where V denotes the potential energy of the system at time t, and E denotes the total energy of the system when $t = 0$.

Solution by R. Holmes, B.A.; Belle Easton, B.A.; *and others.*

Equation of motion $mx'' = X - kx'$, &c.

Hence $m(x''x' + y''y' + z''z') = Xx' + Yy' + Zz' - k(x'^2 + y'^2 + z'^2)$;

$\therefore \quad \frac{m}{2}\frac{d}{dt}(x'^2 + y'^2 + z'^2) = Xx' + Yy' + Zz' - \frac{2k}{m} \cdot \tfrac{1}{2}m(x'^2 + y'^2 + z'^2)$.

But $$V = -\int (Xx' + Yy' + Zz')\, dt,$$

therefore $$\frac{dV}{dt} = -(Xx' + Yy' + Zz');$$

also $\tfrac{1}{2}m(x'^2 + y'^2 + z'^2) =$ kinetic energy at time t,
$= \text{total energy} - V = E - D - V$;

therefore $$\frac{d}{dt}(E - D - V) = -\frac{dV}{dt} - \frac{2k}{m}(E - D - V),$$

therefore $$\frac{dD}{dt} + \frac{2k}{m}(D + V - E) = 0.$$

8872. (Professor Ignacio Beyens.)—Si dans un triangle on voit, du centre du cercle inscrit, sous un angle droit, la distance entre l'orthocentre et le circumcentre, démontrez que $a^2 + b^2 + c^2 = 4(2R - r)(R + r)$.

Solution by R. Knowles, B.A.; J. O'Regan; *and others.*

Let I, C, O be the in-, circum-, and ortho-centres respectively; then we have $CI^2 = R^2 - 2Rr$, $IO^2 = 2r^2 - 4R^2 \cos A \cdot \cos B \cdot \cos C$,

$$CO^2 = R^2(1 - 8\cos A \cdot \cos B \cdot \cos C),$$

and by the question $$CI^2 + IO^2 = CO^2,$$

therefore $$2R^2 \cos A \cdot \cos B \cdot \cos C = rR - r^2.$$

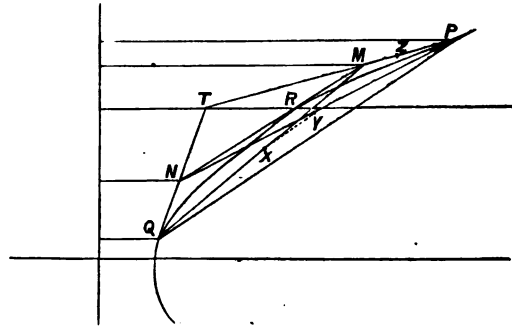
Now, $$1 + \cos A \cdot \cos B \cdot \cos C = (a^2 + b^2 + c^2) / 8R^2;$$

$\therefore \quad 2R^2 \cos A \cdot \cos B \cdot \cos C = \frac{1}{4}(a^2 + b^2 + c^2) - 2R^2 = Rr - r^2,$

whence $$a^2 + b^2 + c^2 = 4(2R - r)(R + r).$$

9068. (R. Knowles, B.A.)—Tangents TP, TQ meet a parabola in P and Q; a tangent at R, the point where the diameter through T meets the curve, meets TP, TQ in M and N respectively; prove that the line joining the mid-points of MQ, NP is parallel to PQ.

Solution by J. M. Stoops, M.A.; Professor Beyens; *and others.*



Let X, Y, Z be the middle points of MQ, NP, MP. From P, M, T, N, Q draw perpendiculars to directrix. Because MP, MR have equal projections on the directrix, therefore MP = MT, so NQ = NT; therefore MN is parallel to PQ. Also XZ is parallel to PQ; and YZ is parallel to MN, and therefore to PQ. Thus XYZ is a straight line, and XY is parallel to PQ.

9095. (R. Holmes, B.A.)—On a given semicircle ACB, take any point Q, and draw QX perpendicular to AB; then construct a triangle on AB with its vertex in ACB such that the in-centre is on QX.

Solution by D. Biddle; Sarah Marks, B.Sc.; *and others.*

Let AX = a, XY = y; then

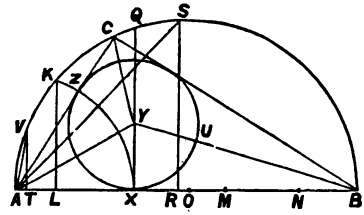
$$y/a = \tan \mathrm{YAX},$$

and $y/(1-a) = \tan \mathrm{YBX}$.

But $\tan 2\mathrm{YAX} = \cot 2\mathrm{YBX}$,

therefore

$$\frac{2y}{a} \Big/ \left(1 - \frac{y^2}{a^2}\right) = \left\{1 - \frac{y^2}{(1-a)^2}\right\} \Big/ \frac{2y}{1-a},$$



whence $$y^2 = \tfrac{1}{2} + a - a^2 \pm (\tfrac{1}{4} + a - a^2)^{\frac{1}{2}}.$$

Such being the case, with centre A describe the arc XK, and draw the perpendicular KL. Then AL = a^2, and LX = $a - a^2$. Make XM = $\frac{1}{2}$AB,

and $XN = \frac{1}{2}AB$. Then $LM = \frac{1}{4} + a - a^2$, and $LN = \frac{1}{4} + a - a^2$. Make $AR = LM$, and draw RS at right angles to AB. Join AS, and make

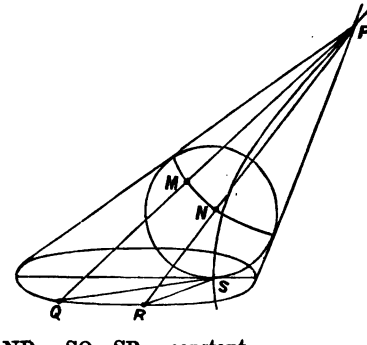
$$AT = LN - AS = \tfrac{1}{4} + a - a^2 - (\tfrac{1}{4} + a - a^2)^{\frac{1}{2}} = y^2.$$

Draw TV at right angles to AB, and make $XY = AV = y$. With centre Y, and radius YX, describe the circle XZU. Tangents to this from A and B will meet at C, on the semicircle drawn on AB, and ABC will be the triangle required.

9101. (W. S. B. Woolhouse, F.R.A.S.)—Let a hyperbola and an ellipse have their planes at right angles, and be such that the vertices of each shall be the foci of the other. Then if, as poles, any two points be taken, one in each branch of the hyperbola, and from them radii be drawn to the ellipse, the sum of each pair of radii will be constant for all points in the ellipse. If the two arbitrary points be taken in the same branch of the hyperbola, the difference of the radii will then be constant in value. It will also follow that, if any two poles be taken in the periphery of the ellipse, and from them radii drawn to the hyperbola, the difference of the radii will again be constant for all points in the hyperbola.

Solution by Alfred Harker, M.A.

It is evident that the cone having the ellipse for base, and any point P on the hyperbola for vertex, is a circular cone, and therefore a sphere can be inscribed in it touching the base in a point S which will be a focus of the ellipse. If Q, R be fixed points on the ellipse, and PQ, PR be drawn meeting in M, N the circle of contact of the sphere and cone, then, since tangents from the same point to a sphere are equal, PM = PN, MQ = SQ, and NR = SR. Hence



$$PQ - PR = MQ - NR = SQ - SR = \text{constant}.$$

The other parts of the question may be solved by similar constructions. [See *Mathematical Tripos*, for 1887, last question on morning of May 18.]

8913. (Professor Hudson, M.A.)—If $\sin(x + h) = \sin x + h \cos(x + \theta h)$, prove that the coefficient of h^3 in the expansion of θ in powers of h is

$$\tfrac{1}{5760}\{57 \cot x + 55 \cot^3 x\}.$$

Solution by R. F. Davis, M.A.

Here $\sin(x+h) = \sin x + h\cos(x+\theta h)$, and it is required to expand θ. in powers of h. Assume $\theta = a + bh + ch^2 + dh^3 + \ldots$

Expanding, by Taylor's theorem,

$$\sin x + h\cos x - \tfrac{1}{2}h^2\sin x - \tfrac{1}{6}h^3\cos x + \tfrac{1}{24}h^4\sin x + \tfrac{1}{120}h^5\cos x + \ldots$$

$$= \sin x + h\left\{\cos x - \theta h\sin x - \tfrac{1}{2}\theta^2h^2\cos x + \tfrac{1}{6}\theta^3h^3\sin x + \tfrac{1}{24}\theta^4h^4\cos x \ldots\right\}$$

$$= \sin x + h\cos x - h^2\sin x\left\{a + bh + ch^2 + dh^3 + \ldots\right\}$$

$$- \tfrac{1}{2}h^3\cos x\left\{a^2 + 2abh + (b^2 + 2ac)\,h^2 + \ldots\right\}$$

$$+ \tfrac{1}{6}h^4\sin x\left\{a^3 + 3a^2bh + \ldots\right\} + \tfrac{1}{24}h^5\cos x\left\{a^4 + \ldots\right\}.$$

Equating coefficients of like powers of h, we get

$$a = \tfrac{1}{2},\ b = \tfrac{1}{24}\cot x,\ c = -\tfrac{1}{48}(1 + \cot^2 x),\ d = \tfrac{1}{5760}\left\{57\cot x + 55\cot^3 x\right\}.$$

9067. (F. Morley, M.A.)—Three particles are connected by two inelastic threads of any length, and placed on a smooth table in a line perpendicular to opposite edges. The outer two, which are of equal mass, are drawn just over the edges and let fall simultaneously. Show that, supposing the inner particle not to leave the table, after the second chuck, all is still.

Solution by D. Biddle.

Let A, B be the falling particles; C the particle on the table. The rationale of absolute rest, when the second thread becomes taut, is that gravity has developed equal momenta in opposite directions, having had A and B (of equal mass M) under its influence during an identical period. Let the thread connecting A with C be the first to become taut, in a time t_1; then the velocity attained at that instant by A $= t_1 g$. B's velocity at the same instant is identical. Subsequently, since A is fettered by its attachment to C, its velocity is to that of B as A/(A + C) : 1. But the momentum on both sides remains unaffected by this circumstance. If t_2 represent the time during which B falls from rest, until its thread becomes taut, its latest velocity $= t_2 g$, and its momentum $= t_2 g$M. The momentum of A just before C begins to move $= t_1 g$M. The additional momentum gained between the two "chucks" $= (t_2 - t_1)\,g$M. The momentum has increased at the same rate as before, the velocity only being altered. Therefore equilibrium instantly occurs when the tightening of the connecting threads brings the equal forces into opposition.

8822. (Professor Sylvester, F.R.S.) — Let U be any cubic surface, Θ the covariant to U of the 9th order in the variables and 11th degree in the coefficients which contains the 27 ridges of U. Prove that $x^6\Theta - 2x^3\mathrm{U}\frac{d}{da}\Theta + \mathrm{U}^2\frac{d^2}{da^2}\Theta$ is a scroll which contains the 27 ridges of $\mathrm{U} + \lambda x^3$ for every value of λ.

Solution by the PROPOSER.

Let the two circular points of infinity have for direct coordinates α, β, γ; α', β', γ', then the equations to these two points taken separately will be $a\alpha A + b\beta B + c\gamma C = 0$, $a\alpha' A + b\beta' B + c\gamma' C = 0$,

and their collective equation will be

$$\Sigma \left\{ a^2 \alpha\alpha' A^2 + bc\,(\beta\gamma' + \beta'\gamma)\, BC \right\} = 0.$$

But the circular points at infinity may be regarded as the intersections of the line at infinity $a\alpha + b\beta + c\gamma = 0$ with the circle circumscribed about the triangle of reference $a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$, between which two equations, eliminating γ, we obtain $(a\alpha + b\beta)(a\beta + b\alpha) - c^2\alpha\beta = 0$,

or $$ab\alpha^2 - (c^2 - a^2 - b^2)\,\alpha\beta + ab\beta^2 = 0.$$

Hence $$\frac{\alpha}{\beta} - \frac{\alpha'}{\beta'} = 1, \text{ and } \frac{\alpha}{\beta} + \frac{\alpha'}{\beta'} = \frac{c^2 - a^2 - b^2}{ab};$$

and, as we may obtain other similar relations to these by eliminating β instead of γ, there results

$$\alpha\alpha' : \beta\beta' : \gamma\gamma' : \beta\gamma' + \beta'\gamma : \gamma\alpha' + \gamma'\alpha : \alpha\beta' + \alpha'\beta$$
$$= 1 : 1 : 1 : a^2 - b^2 - c^2 : b^2 - c^2 - a^2 : c^2 - a^2 - b^2.$$

Hence $$\Sigma \left\{ a^2 \alpha\alpha' A^2 + bc\,(\beta\gamma' + \beta'\gamma)\, BC \right\} = \Sigma \left\{ a^2 A^2 + (a^2 - b^2 - c^2)\, BC \right\}$$
$$= \Sigma a^2\,(A - B)\,(A - C),$$

and the collective equation to the two circular points at infinity is

$$\Sigma a^2\,(A - B)\,(A - C) = 0, \text{ as was to be shown.}$$

Observation.—We know that for all finite values of A, B, C,

$$\Sigma a^2\,(A - B)\,(A - C) = E^2,$$

where E is twice the area of the triangle of reference. Hence it is easily seen that the above equation expresses the fact that, if A, B are any two points on the finite region of the plane, and I is either point of the absolute, $\sin IAB = \infty$.

9036. (Professor HARKOMA, M.A.)—A straight rod is divided at random into three parts; prove that the chance that twice the sum of the squares on the parts shall be less than the square on the whole rod is $\pi / (3\sqrt{3})$.

Solution by F. X. DE WACHTER; Prof. NASH, M.A.; *and others.*

By geometry, if we construct the equilateral triangle whose height $= l$ = length of the rod, the sum of the distances of any interior point to the three sides $= l$. Moreover, any circle being described having its centre in that of the triangle, the sum of the squares of the above distances, for any point of the circumference, will be constant. If that circle be inscribed to the triangle, the sum of the squares $= \frac{1}{2}l^2$, while, for any point within that inscribed circle, the sum of squares $< \frac{1}{2}l^2$, and for any point without, the sum of squares $> \frac{1}{2}l^2$.

The probability is, therefore, the quotient of the areas

$$\textit{circle} : \textit{triangle} = \pi : 3\sqrt{3}.$$

9111. (B. HANUMANTA RAU, B.A.)—Solve in *positive* integers the equation $x^2-19y^2=81$.....(1).

Solution by H. PLAMENEWSKY; H. L. ORCHARD, B.Sc., M.A.; *and others.*

1. x and y prime to one another: $(x+9)(x-9)=19y^2$, $x\pm9=19n$.

2. $n(19n\mp18)=y^2$, $x=19n\mp9$:

(A) $n=0$, $y_0=0$, $x_0=9$, $n=1$, $y_1=1$, $x_1=10$,

(B) $n=\alpha\beta^2$, $y=\alpha\beta\gamma$, $19\beta^2+\frac{18}{\alpha}=\gamma^2$.....(2).

$\alpha=2$: $\gamma^2-19\beta^2=9$, $(\gamma+3)(\gamma-3)=19\beta^2$, $\gamma\pm3=19k$, $k^2\left(19\pm\frac{6}{k}\right)=\beta^2$,

$k=1$, $\beta_1=5$, $\gamma_1=22$, $k=2$, $\beta_2=8$, $\gamma_2=35$,

$x_2=959$, $y_2=220$, $x_3=2441$, $y_3=560$.

(C) $x=3\gamma$, $y=3\beta$: $\gamma^2-19\beta^2=9$ .....(3),

$x_4=3\cdot22=66$, $y_4=3\cdot5=15$, $x_5=3\cdot35=105$, $y_5=3\cdot8=24$.

(D) $x=9z$, $y=9t$: $z^2-19t^2=1$..... .....(4).

For every equation $x^2-ay^2=1$, $x=2a^2-1$, $y=2a\beta$, where a and β are one of the solutions, we have

$$4+\cfrac{1}{2+\cfrac{1}{1\ldots}}<\frac{170}{39},\quad a_1=170,\ \beta_1=39,$$

$z_1=170$, $z_2=57799$, $z_3=57799^2\cdot2-1$, $t_1=39$, $t_2=13260$, $t_3=2\cdot z_2t_2$.

Whence $x_6=170\cdot9=1530$, $y_6=39\cdot3=117$, $x_7=510191$, $y_7=119340$,

$$\ldots\ldots x_n=\frac{2x_{n-1}^2}{9}-9,\quad y_{12}=\frac{2x_{n-1}\cdot y_{n-1}}{9}\ldots\ldots$$

[We have $(x+9)(x-9)=19y^2$, and, if $y=1$, then $x=10$ is an obvious solution. Again, $\sqrt{19}=4+\cfrac{3}{8+\frac{3}{8}}$ + &c. The convergents are

$$\tfrac{4}{1},\ \tfrac{33}{8},\ \tfrac{292}{87},\ \tfrac{2441}{560},\ \tfrac{20101}{4681},\ \text{&c.}$$

The n^{th} convergent will solve the equation $x^2-19y^2=(-1)^n\cdot3^n$; therefore $x=2441$, $y=560$ is another solution.]

9112. (Professor SYLVESTER, F.R.S.)—Prove that a number may be represented by a sequence of numbers (a single number by itself counting as a sequence) in as many ways as it contains odd factors. (*Ex.*—The odd factors of 20 are two in number, 1 and 5; and 20 is equal to the following sequences, viz., $2+3+4+5+6$, and 20.) Again, the odd factors of 45 are six in number, viz., 1, 3, 5, 9, 15, 45, and 45 may be represented by the six sequences (and no others),

$$1+2+3+4+5+6+7+8+9,\quad 5+6+7+8+9+10,\quad 7+8+9+10+11,$$
$$14+15+16,\quad 22+23,\quad 45.$$

Solutions by (1) Prof. CULLEY, M.A.; (2) R. W. D. CHRISTIE, M.A.

1. The sum of a sequence of x numbers, commencing with y, is $\frac{1}{2}x(2y+x-1)$. We require, therefore, to find the number of solutions of the equation $\frac{1}{2}x(2y+x-1) = N \equiv 1 \cdot 2^p \cdot b^q \cdot c^r \ldots\ldots$,

where y must be an integer, and x a positive integer. Now, when $2y+x-1$ is odd, x is even, and *vice versâ* (1). If we put $2y+x-1 = b^{q-k}c^{r-l}\ldots$, *i.e.*, to any *odd* factor of N, then $x = 2^{p+1} \cdot b^k c^l \ldots$, which is an even integer. If, however, we put $2y+x-1 = 2^{p-m} \cdot b^{q-k} \cdot c^{r-l}\ldots$, *i.e.*, to any *even* factor of N, then we must have $x = 2^{m+1} b^k c^l$, which is even, contrary to (1), and therefore no solution. There is no objection to negative values of y; *e.g.*, put $2y+x-1 =$ the odd factor 1; then $x = 2N$, and $y = -(N-1)$; this sequence of 2N terms, starting from $-(N-1)$, and proceeding to $+N$, is clearly equivalent to the sequence N. Therefore for every odd factor of N we obtain one sequence, and these are all.

2. *Otherwise*:—Let

$$N = (a) + (a+1) + (a+2) \ldots\ldots (a+n-1) = n\left\{a + \tfrac{1}{2}(n-1)\right\},$$

Also $N = (1-a) + (2-a) + (3-a) \ldots\ldots (n-a) = m\left\{(1-a) + \tfrac{1}{2}(m-1)\right\}$

$$= 2n_1\left\{(1-a) + \tfrac{1}{2}(2n_1-1)\right\}, \text{ say,}$$

where a is an integer. Then we have

$$N = n\left\{a + \tfrac{1}{2}(n-1)\right\} = 2n_1\left\{(1-a) + \tfrac{1}{2}(2n_1-1)\right\}.$$

If we suppose $nn_1 = N$, we have the lowest and highest factors of N. (1) Let n be *even*; then both (a) and $(1-a)$ are *fractional*. (2) When n is *odd*, a becomes negative as n increases, but when a is *negative* $(1-a)$ is *positive*. Thus we can always secure a sequence without negative numbers with *odd* factors.

Examples: $n = 9$, $n_1 = 7$, and $n = 21$, $n_1 = 3$.

Then $N = 63 = 9\left\{3+4\right\} = \left\{-2 + \frac{13}{2}\right\} 14$,

and we have the positive sequence $3+4+5+6+7+8+9+10+11$;

and $N = 63 = 21\left\{-7+10\right\} = \left\{8 + \frac{5}{2}\right\} 6$,

and we have the positive sequence $8+9+10+11+12+13$.

[The question may be extended as follows:—"Any number may be represented by a series whose common difference is 2, in as many ways (and no more) as there are factors not greater than its square root."]

9102. (H. L. ORCHARD, B.Sc., M.A.)—Show that the series $1^7 + 2^7 + 3^7 + 4^7 + \ldots + 9^7$ is divisible by 27.

Solution by H. FORTEY, M.A.; R. KNOWLES, B.A.; *and many others.*

Here 3^7, 6^7, 9^7 are each $= 0$ (mod. 27).

Also, to the same modulus,

$$1^7 = 1,\ 2^7 = -7,\ 4^7 = -5,\ 5^7 = -13,\ 7^7 = 16,\ 8^7 = 8;$$

hence $1^7 + 2^7 + 3^7 + \ldots + 9^7 = 1-7-5-13+16+8 = 0$ (mod. 27).

8702. (R. KNOWLES, B.A.)—Prove that

$$\int x^{n-1}\log(1+x)\,dx = \frac{1}{n}\left(x^n\int\frac{dx}{1+x} - \int\frac{x^n dx}{1+x}\right).$$

Solution by Rev. T. R. TERRY, M.A.; G. G. STORR, M.A.; *and others.*

Since $$\int\frac{dx}{1+x} = \log(1+x)$$

we have differential of right-hand side $= x^{n-1}\log(1+x)$, which proves the required result.

8911. (Professor BYOMAKESA CHAKRAVARTI, M.A.)—The tangents at A, A′, extremities of a diameter of an ellipse, meet the tangent at P in Y, Y′; CB is the semi-diameter conjugate to CA; prove that the areas of the triangles YCY′, ABA′, APA′ are in geometrical progression.

Solution by C. E. WILLIAMS, M.A.; R. KNOWLES, B.A.; *and others.*

Let YY′ meet CB produced at T.

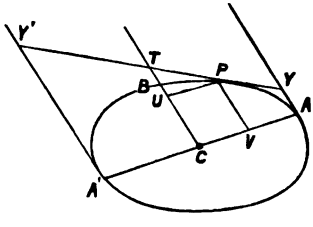
Then YCY′ = ATA′,

and the triangles ATA′, ABA′, APA′, being on the same base AA′, are as their altitudes, and therefore as

CT : CB : CU,

and we know that

$CT \cdot CU = CB^2$.



9080. (Professor NEUBERG.)—Soient A_n le centre de gravité des points $A_0, A_1, \ldots A_{n-1}$ chargés des masses $a_0, a_1 \ldots a_{n-1}$; A_{n+1} le centre de gravité des points $A_1, A_2, \ldots A_{n-1}, A_n$ chargés des masses $a_0, a_1 \ldots a_{n-2}, a_{n-1}$; A_{n+2} celui des points $A_2, A_3 \ldots A_{n+1}$ chargés respectivement des mêmes masses $a_0, a_1 \ldots a_{n-1}$; et ainsi de suite. Trouver la limite des points $A_n, A_{n+1} \ldots$

Solution by (I.) A. R. JOHNSON, M.A.; (II.) *the* PROPOSER.

I. Use A_m to denote the distance of A_m from a fixed axis. Then

$$(a_0+a_1+\ldots+a_{n-1})A_m - a_{n-1}A_{m-1} - a_{n-2}A_{m-2} - \ldots - a_0 A_{m-n} = 0.$$

Hence $$A_m = B + \sum_{r=1}^{r=n-1} C_r x_r^m,$$

where x_r $(r = 1, 2, \ldots n-1)$ are the roots other than unity of

$$f(x) \equiv (a_0 + a_1 + \ldots + a_{n-1})\, x^n - a_{n-1}x^{n-1} - \ldots - a_0 = 0,$$

and the C's are constants. When this tends to a limit, each of the x's must be < 1, and the limit is B.

Eliminating the C's from the system

$$A_s = B + \sum_{r=1}^{r=n-1} C_r x_r^s, \quad (s = 0, 1, \ldots n-1),$$

we have, denoting $\frac{f(x)}{x-1}$ by $p_{n-1}x^{n-1} + p_{n-2}x^{n-2} + \ldots + p_0$,

$$B \sum_{s=0}^{s=n-1} p_s = \sum_{s=0}^{s=n-1} p_s A_s,$$

as appears by multiplying A_s by p_s and summing, when the C's disappear since the x's are roots of $\frac{f(x)}{x-1} = 0$.

But $p_s = \sum_{r=0}^{r=s} a_r$. Hence the limit of the C. G. is the C. G. of the mass-system $\sum_{r=0}^{r=s} a_r$ at A_s $(s = 0, 1, 2, \ldots n-1)$.

II. *Otherwise:*—1. Les masses a étant supposées *positives*, les points A_n, A_{n+1}, A_{n+2}, ... doivent tendre vers un point-limite L. Car, si B_0, B_1, ... sont les projections des points A_0, A_1, ... sur un axe quelconque δ, B_n tombe entre les points extrêmes du groupe $(B_0, B_1, \ldots, B_{n-1})$; B_{n+1} tombe entre les extrêmes du groupe $(B_1, B_2, \ldots, B_n)$, etc. Or les points extrêmes d'un groupe finissent par ne plus se retrouver dans les groupes suivants; donc le segment de δ occupé par les points d'un groupe de n points consécutifs, B peut devenir aussi petit qu'on veut.

2. Je dis que le point-limite L est le centre de gravité des points donnés $A_0, A_1, \ldots, A_{n-1}$ chargés des masses suivantes

$$(1)\ldots\ldots \begin{cases} A_0 & \ldots a_0 & & & & \\ A_1 & \ldots a_1 & +a_0 & & & \\ A_2 & \ldots a_2 & +a_1 & +a_0 & & \\ A_3 & \ldots a_3 & +a_2 & +a_1 & +a_0 & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \\ A_{n-1} & \ldots a_{n-1} & +a_{n-2} & +a_{n-3} & +a_{n-4} & +\ldots +a_0 \end{cases}$$

En effet, les masses inscrites dans la première *colonne* peuvent être remplacées par une masse unique $a_0 + a_1 + \ldots + a_{n-1}$ attachée au point A_n; donc le centre de gravité du système (1) est le même que celui des mêmes masses attachées respectivement aux points $A_1, A_2, \ldots A_{n-1}$. En continuant ainsi, on verra que le barycentre du système (1) ne change pas, lorsqu'on remplace les points $A_0, A_1, \ldots A_{n-1}$ par $A_m, A_{m+1}, A_{m+2}, \ldots A_{m+n}$. Mais ces derniers points tendent à se confondre avec L pour m infini; donc, etc., ...

8377. (Asparagus.)—The normal at any point p of a given ellipse meets the axes in G, g; Gg is bisected in O, and from O are drawn three

other normals (always real) OP, OQ, OR; prove that P, Q, R will be the corners of a maximum triangle inscribed in the ellipse, and the circles of curvature at P, Q, R will concur in a point p' of the ellipse such that pp' is a diameter.

Solution by R. F. Davis, M.A.; Prof. Chakravarti, M.A.; *and others.*

Let α be the eccentric angle of p, the normal at which has for its equation $ax\sec\alpha - by\operatorname{cosec}\alpha = c^2$. Then the coordinates of O are $c^2\cos\alpha/2a$, $-c^2\sin\alpha/2b$; and it will lie on the normal at P(ϕ), if $\cos\alpha/2\cos\phi + \sin\alpha/2\sin\phi = 1$, or $\sin(\phi+\alpha) = \sin 2\phi$. Rejecting the root $\phi = \alpha$, we have $3\phi = \pi - \alpha$. Hence the eccentric angles of P, Q, R are $\frac{1}{3}(\pi-\alpha)$, $\frac{1}{3}(3\pi-\alpha)$, $\frac{1}{3}(5\pi-\alpha)$, and correspond to the vertices of an equilateral (maximum) triangle inscribed in the auxiliary circle. The circles of curvature at P, Q, R cointersect in a point on the curve whose eccentric angle $= -(\pi-\alpha)$, or p' (Salmon.)

9104. (Prof. Mahendra Nath Ray, M.A., LL.B.)—Show that

$$\frac{\cos\theta}{1-\sin\theta} + \frac{\cos\phi}{1-\sin\phi} = \frac{2(\sin\theta-\sin\phi)}{\sin(\theta-\phi)+\cos\theta-\cos\phi}.$$

Solution by G. G. Storr, M.A.; Sarah Marks, B.Sc.; *and others.*

Since $\sin(\theta-\phi)+\cos\theta-\cos\phi = \cos\theta(1-\sin\phi)-\cos\phi(1-\sin\theta)$, we have but to show that

$\cos^2\theta(1-\sin\phi)^2-\cos^2\phi(1-\sin\theta)^2 = 2(1-\sin\theta)(1-\sin\phi)(\sin\theta-\sin\phi)$, which, by expressing the cosines in terms of the sines, is at once seen to be true.

9096. (M. F. J. Mann, B.A.)—If $n = 2^K\lambda^\alpha\mu^\beta\ldots$, and N does not contain as a factor a power of a number a higher power of which is a factor of n, prove that $N^{mL+1}-N$ is a multiple of n, where L = L. C. M. of $2^{K-2}, \lambda^{\alpha-1}, \mu^{\beta-1}, \ldots (\lambda-1), (\mu-1)\ldots$, and m is any integer.

8899. (B. Hanumanta Rau, B.A.)—If N is any number in the scale whose radix is 13, 14, or 15, N ends in the same digit as N^{13}.

Solution by M. F. J. Mann, B.A.

(9096.) Let $N = q.\lambda^\alpha\ldots$, where q is prime to n; then N is prime to $2^K\mu^\beta\ldots$. The number of values of x less than $2^K\mu^\beta\ldots$ satisfying $x^L = M(2^K\mu^\beta\ldots)+1$ [by Quest. 9058] is $2^{K-1}\mu^{\beta-1}\ldots(\mu-1)\ldots$, which is the number of numbers less than $2^K\mu^\beta\ldots$ and prime to it; hence any value of x prime to $2^K\mu^\beta\ldots$ satisfies $x^L = M(2^K\mu^\beta\ldots)+1$, therefore

$$N^L = M(2^K\mu^\beta\ldots)+1, \text{ therefore } N^{mL} = M(2^K\mu^\beta\ldots)+1,$$

therefore $N^{mL+1} = M(2^K\mu^\beta\ldots)q\lambda^\alpha\ldots+N$, therefore $N^{mL+1}-N = M(n)$.

(8899.) This is a particular case of the above Ques. 9096.

9155. (Professor CROFTON, F.R.S.)—Find an operator ζ which satisfies the condition $\zeta\chi - \chi\zeta \equiv 2\mathrm{D}$, where χ is a given function of x, and $\mathrm{D} = d/dx$.

Solution by the PROPOSER.

Differentiating as to D, we have

$$\frac{d\zeta}{d\mathrm{D}}\chi - \chi\frac{d\zeta}{d\mathrm{D}} \equiv 2, \quad \frac{d^2\zeta}{d\mathrm{D}^2}\chi - \chi\frac{d^2\zeta}{d\mathrm{D}^2} \equiv 0 \quad \ldots\ldots\ldots\ldots(1, 2).$$

Hence $\frac{d^2\zeta}{d\mathrm{D}^2}$ cannot contain D; hence (calling α, β, γ ... arbitrary functions of x), $\frac{d^2\zeta}{d\mathrm{D}^2} = \alpha$; and hence we may either conclude

$$\frac{d\zeta}{d\mathrm{D}} = \alpha\mathrm{D} + \beta, \quad \text{or} \quad \mathrm{D}\alpha + \beta,$$

and we may conclude $\zeta = \mathrm{D}\alpha\mathrm{D} + \beta\mathrm{D} + \gamma$ .....(3).

[α, β are not quite the same as the former, but it is easy to see the most general form for ζ is as given, or can be so transformed.]

To determine the arbitrary functions α, β, substitute this value of ζ in (1), $(\mathrm{D}\alpha + \alpha\mathrm{D} + \beta)\chi - \chi(\mathrm{D}\alpha + \alpha\mathrm{D} + \beta) = 2$.

Now, since $\mathrm{D}\chi - \chi\mathrm{D} \equiv \chi'$, therefore $\chi'\alpha + \alpha\chi' = 2$,

therefore $$\alpha = \frac{1}{\chi'} \quad \ldots\ldots\ldots\ldots(4).$$

Now, it is easily seen that

$$\mathrm{D}\alpha\mathrm{D}\chi - \chi\mathrm{D}\alpha\mathrm{D} \equiv \mathrm{D}\alpha\chi' + \alpha\chi'\mathrm{D} = 2\mathrm{D}, \text{ by (4).}$$

Hence, if we substitute the value (3) for ζ in the original equation,

$$\beta\mathrm{D}\chi - \chi\beta\mathrm{D} \equiv 0, \quad \text{hence } \beta = 0.$$

Hence the value required is $\zeta \equiv \mathrm{D}\frac{1}{\chi'}\mathrm{D} + \gamma$, γ being any function of x. This may be easily verified.

9168. (The EDITOR.)—Show that one root of the equations $z^2 + 10x = 5(y^3 - 7x^2) = 20x^2$, $y + z = 28$, is $x = 4{\cdot}510664$, and ascertain the nature of the other roots.

Solution by D. BIDDLE.

We readily find $y = (7x^3 + 4x^2)^{\frac{1}{3}}$ and $z = (20x^2 - 10x)^{\frac{1}{2}}$. Therefore we have $(7x^3 + 4x^2)^{\frac{1}{3}} + (20x^2 - 10x)^{\frac{1}{2}} = 28$. This, in the ordinary way, resolves itself into an equation of the 6th degree; but the required root can be found by the following method:—Let $v = 2/x$, then

$$(7 + 2v)^{\frac{1}{3}} + (20 - 5v)^{\frac{1}{2}} = 14v \quad \ldots\ldots\ldots\ldots(\mathrm{A}).$$

Suppose $v = 0$, then we must affix $(+6{\cdot}3851)$ to the right side of the equation, to counterbalance the error. Suppose $v = 1$, then $(-8{\cdot}0469)$ is in like manner required. By the difference in sign of the two appendices,

we know that the actual value of v lies between the two supposed numbers, 0 and 1; and $(6\cdot3851)/(6\cdot3851+8\cdot0469)$ gives the first figure of it, namely, $\cdot4$.

Acting similarly, suppose

$v = \cdot4$,	then $(7+2v)^{\frac{1}{2}}+(20-5v)^{\frac{1}{2}}$	$=$	$14v+0\cdot6258$
$v = \cdot5$,	,,	,,	$= 14v-0\cdot8167$
$v = \cdot44$,	,,	,,	$= 14v+0\cdot0490$
$v = \cdot45$,	,,	,,	$= 14v-0\cdot0953$
$v = \cdot443$,	,,	,,	$= 14v+0\cdot0057$
$v = \cdot444$,	,,	,,	$= 14v-0\cdot0088$
$v = \cdot4433$,	,,	,,	$= 14v+0\cdot0013$
$v = \cdot4434$,	,,	,,	$= 14v-0\cdot0001$
$v = \cdot44339$,	,,	,,	$= 14v+0\cdot000055$
$v = \cdot443393$,	,,	,,	$= 14v+0\cdot000011$
$v = \cdot443394$,	,,	,,	$= 14v-0\cdot000004$

therefore $v = \cdot4433937$, and $2/v = x = 4\cdot510664$, whence $y = 8\cdot978570$ and $z = 19\cdot021430$. These are the only real roots, since it is easy to see from (A) that, if v be supposed negative, the right side of the equation will be negative, but the second and greater term of the left side always positive.

8279. (Professor WOLSTENHOLME, Sc.D.) — A common tangent to two conics S, S′ touches them in A, A′, and through a fixed point H on this common tangent is drawn any straight line meeting the two conics in P, Q; P′, Q′, respectively: prove that the points of intersection of AP, AQ with A′P′, A′Q′ lie on a fixed conic passing through the four common points of S, S′.

Solution by the PROPOSER.

Let another common tangent BB′ meet the former in O, and, referring to these as coordinate axes, take

$$S \equiv \left(\frac{x}{a}+\frac{y}{b}-1\right)^2+2\lambda\frac{xy}{ab},\quad S' = \left(\frac{x}{a'}+\frac{y}{b'}-1\right)^2+2\lambda'\frac{xy}{a'b'},$$

so that $OA = a$, $OA' = a'$; also let $OH = h$, then the equations of AP, A′P′ may be taken to be $my = x-a$, $m'y = x-a'$; and, if (x_1y_1), (x_2y_2) be the points P, P′, we must have

$$h(y_1-y_2) = x_2y_1-x_1y_2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)},$$

since H, P, P′ are collinear.

Now, $my_1 = x_1-a$, and $\left(\frac{x_1}{a}+\frac{y_1}{b}-1\right)^2+2\lambda x_1\frac{y_1}{ab} = 0$, whence

$$y_1\left(\frac{m}{a}+\frac{1}{b}\right)^2+2\lambda\frac{x_1}{ab} = 0,$$

or
$$\frac{x_1}{\left(\frac{m}{a}+\frac{1}{b}\right)^2} = \frac{y_1}{-\frac{2\lambda}{ab}} = \frac{x_1-my_1}{\left(\frac{m}{a}+\frac{1}{b}\right)^2+\frac{2\lambda m}{ab}} = \frac{a}{\left(\frac{m}{a}+\frac{1}{b}\right)^2+\frac{2\lambda m}{ab}};$$

and, similarly, $$\frac{x_2}{\left(\frac{m'}{a'}+\frac{1}{b'}\right)^2}=\frac{y_2}{-\frac{2\lambda'}{a'b'}}=\frac{a'}{\left(\frac{m'}{a'}+\frac{1}{b'}\right)^2+\frac{2\lambda'm'}{a'b'}};$$

whence, substituting in (A),

$$h\left\{\frac{\lambda'}{b'}\left[\left(\frac{m}{a}+\frac{1}{b}\right)^2+\frac{2\lambda m}{ab}\right]-\frac{\lambda}{b}\left[\left(\frac{m'}{a'}+\frac{1}{b'}\right)^2+\frac{2\lambda'm'}{a'b'}\right]\right\}$$
$$=\frac{\lambda'a}{b'}\left(\frac{m}{a}+\frac{1}{b}\right)^2-\frac{\lambda a'}{b}\left(\frac{m'}{a'}+\frac{1}{b'}\right)^2;$$

or $$(h-a)\frac{\lambda'}{b'}\left(\frac{m}{a}+\frac{1}{b}\right)^2-(h-a')\frac{\lambda}{b}\left(\frac{m'}{a'}+\frac{1}{b'}\right)^2=\frac{2\lambda\lambda'}{bb'}h\left(\frac{m'}{a'}-\frac{m}{a}\right)$$
.....(B).

But, if (x, y) be the point of intersection of AP, A'P', we have

$$m=\frac{x-a}{y},\quad m'=\frac{x-a'}{y},$$

and, substituting these values in (B), we get the following as the equation of the required locus

$$\frac{\lambda'}{b'}(h-a)\left(\frac{x}{a}+\frac{y}{b}-1\right)^2-\frac{\lambda}{b}(h-a')\left(\frac{x}{a'}+\frac{y}{b'}-1\right)^2=\frac{2\lambda\lambda'}{bb'}xy\left(\frac{h}{a'}-\frac{h}{a}\right),$$

or $$\frac{\lambda'}{b'}(h-a)\left(S-\frac{2\lambda xy}{ab}\right)-\frac{\lambda}{b}(h-a')\left(S-\frac{2\lambda'x'y'}{a'b'}\right)=\frac{2\lambda\lambda'}{bb'}xy\left(\frac{h}{a'}-\frac{h}{a}\right),$$

that is, $$\frac{b(h-a)}{\lambda}S=\frac{b'(h-a')}{\lambda'}S',$$

or the locus is a conic passing through the common points of S, S'. (It is obvious that these are points on the locus.) When H coincides with O, $h=0$, and the conic degenerates into that pair of the common chords of S, S' which pass through the common point of AB, A'B'. So, also, if H be either of the points where the remaining common tangents meet AA'. Similarly, if we take a fixed point K on BB' we should obtain the locus $$\frac{a(k-b)}{\lambda}S=\frac{a'(k-b)}{\lambda'}S',$$

where $k=$ OK, and this will coincide with the former if

$$\{OAA'H\}=\{OBB'K\},$$

and there will be one point on each of the other two common tangents, corresponding to H, which will give the same conic.

[*Otherwise*:—Let $x=0$ be the common tangent, (0, 0) the fixed point H, and

$$x(lx+my)-(ax+by+cz)^2\equiv S=0,\quad x(l'x+m'y)-(a'x+b'y+c'z)^2\equiv S'=0$$

the two conics. Now, if $x=\mu y$ be any line through H, at the points P and Q where it meets $S=0$, we have

$$(ax+by+cz)^2=x^2\left(l+\frac{m}{\mu}\right),$$

and, at P' and Q', $$(a'x+b'y+c'z)^2=x^2\left(l'+\frac{m'}{\mu}\right);$$

and these are the equations to the pairs of lines AP, AQ, and A'P', A'Q', and at the intersections

$$m'(ax+by+cz)^2 - m(a'x+b'y+c'z)^2 = x^2(lm'-l'm).$$

Now this is the equation to a conic through the intersections of S and S', with respect to which the triangle, whose vertices are at A, A', and the intersection of the polars of H with respect to the given conics, is self-conjugate.]

9083. (Professor KALIPADA BASU, M.A.)—Solve the equation

$$\cos^{-1}\left(1-\frac{a}{x}\right) = \frac{2a}{(2ax-a^2)^{\frac{1}{2}}}.$$

Solution by MUNINDRANATH BHATTACHARYYA, M.A.

Let $\frac{a}{x} = 1-\cos\theta$, *i.e.*, $x = \frac{a}{1-\cos\theta}$, then the equation becomes $\frac{1}{2}\theta = \tan\frac{1}{2}\theta$, which gives $\frac{1}{2}\theta = 0$, whence $x = \infty$.

[If we take the general value of $\cos^{-1}(\cos\theta)$, we get $2n\pi \pm \theta = 2\tan\frac{1}{2}\theta$, therefore $\frac{1}{2}\theta = 0$ is one solution of this equation, that is when $n = 0$.]

8918. (Prof. MAHENDRA NATH RAY, M.A., LL.B.)—If

$A^{\frac{1}{2}} = (A-2ax-ay+a^2)^{\frac{1}{2}} + (A-ax-2ay+a^2)^{\frac{1}{2}}$, where $A = x^2+xy+y^2$; show (1) that $A = a(x+y)$, and (2) give a geometrical interpretation.

Solution by R. F. DAVIS, M.A.; SARAH MARKS, B.Sc.; *and others.*

1. Since $(A-2ax-ay+a^2)-(A-ax-2ay+a^2) = a(y-x)$,

we have, by division, $(\ldots)^{\frac{1}{2}} - (\ldots)^{\frac{1}{2}} = a(y-x)\,A^{-\frac{1}{2}}$;

hence $2(A-2ax-ay+a^2)^{\frac{1}{2}} = \{A+a(y-x)\}\,A^{-\frac{1}{2}}$.

Squaring each side, it will be found that $3\{A-a(x+y)\}^2 = 0$.

2. Let Ox, Oy be oblique axes inclined at an angle of 60°; A, B points (one on each axis) such that $OA = OB = a$. Then, since OAB is an equilateral triangle, if P moves in such a manner that $PO = PA + PB$, its locus will be the circumscribing circle.

9040. (Professor CHASE, LL.D.)—A point O is taken within a triangle ABC; P, Q, R are the centres of the circles circumscribing the triangles BOC, COA, AOB, respectively; determine the position of O that the area PQR may be a minimum.

Solution by Professor NEUBERG.

Soient α, β, γ les angles BOC, COA, AOB, supposés *obtus*, et soient ω, r le centre et le rayon du cercle ABC. Le triangle ωBP donne, les angles ωPB et PωB valant $180°-\alpha$, A : $P\omega = r\sin(\alpha-A)/\sin\alpha$. D'où

$$2PQR = \omega P.\,\omega R.\sin P\omega R + \ldots = r^2\left[\frac{\sin(\alpha-A)\sin(\beta-B)}{\sin\alpha\sin\beta}\sin C + \ldots\right],$$

ou, à cause de $ABC = 2r^2\sin A\sin B\sin C$:

$$4PQR = ABC(\cot A-\cot\alpha)(\cot B-\cot\beta)+\ldots,$$

et, à cause de $\cot A\cot B+\ldots = 1$, $\cot\alpha\cot\beta+\ldots = 1$:

$$4PQR = ABC\{2-\cot\alpha(\cot B+\cot C)-\ldots\}$$
$$= ABC\left(2-\frac{\cot\alpha\sin^2 A+\cot\beta\sin^2 B+\cot\gamma\sin^2 C}{\sin A\sin B\sin C}\right).$$

On a aussi $\alpha+\beta+\gamma = 360°$. Le minimum de PQR exige $d(PQR) = 0$,

ou $$\frac{d(PQR)}{d\alpha}d\alpha+\frac{d(PQR)}{d\beta}d\beta+\frac{d(PQR)}{d\gamma}d\gamma = 0,\quad d\alpha+d\beta+d\gamma = 0;$$

d'où l'on déduit $$\frac{d(PQR)}{d\alpha} = \frac{d(PQR)}{d\beta} = \frac{d(PQR)}{d\gamma},$$

ou $$\frac{\sin A}{\sin\alpha} = \frac{\sin B}{\sin\beta} = \frac{\sin C}{\sin\gamma}.$$

Le point cherché est donc l'orthocentre de ABC, pourvu que les angles A, B, C soient aigus.

8019. (W. J. C. SHARP, M.A.)—If the weights of three bodies whose centres are at the vertices of a triangle are connected by a homogeneous relation of the n^{th} order, prove that (1) the locus of the centroid of the system is a plane n-ic curve; and (2), if those of four bodies, whose centroids occupy the vertices of a tetrahedron, are connected by such a relation, the locus is an n-ic surface.

Solution by B. HANUMANTA RAU, M.A.; J. O'REGAN; *and others.*

Let A, B, C, D represent the weights of the four weights placed at the points A, B, C, D of a tetrahedron; α, β, γ, δ the distances of the centroid of the system from the planes BCD, CDA, DAB, ABC, respectively; then

$$\alpha = \frac{A\,(\text{perp. from A on BCD})}{A+B+C+D},$$

or $$\frac{\alpha(BCD)}{A} = \frac{3\times\text{volume of tetrahedron}}{A+B+C+D},$$

$$\{(BCD) = \Delta BCD\} = \frac{\beta(CDA)}{B} = \frac{\gamma(DAB)}{C} = \frac{\delta(ABC)}{D}.$$

If $\phi(A, B, C, D) = 0$ is the relation between the weights, then

$$A^n\phi\left(1, \frac{B}{A}, \frac{C}{A}, \frac{D}{A}\right) = 0,$$

or $$\phi\{\alpha(BCD), \beta(CDA), \gamma(CDA), \delta(ABC)\} = 0.$$

This being an equation of the n^{th} degree between $\alpha, \beta, \gamma, \delta$, represents an n-ic surface.

[The result will follow more readily by using tetrahedral coordinates, as these are shown in the solution to be proportional to the weights. The same thing is provable in like manner with respect to the Simplicissimum coordinates of the centroid of weights at the vertices of the Simplicissimum of reference (Quest. 8242), and hence the proposition is true in space of any dimensions, including plane space (1)].

8665. (Rev. T. R. Terry, M.A.)—If the conjugate quaternion Kq is defined by the equation $q + Kq = 2Sq$, prove that

(1) $K(qr) = Kr \,.\, Kq$, and (2) $rq + qr + r^{-1}(rq + qr)\, Kr = 4qSr$.

Solution by Prof. Mathews; *the* Proposer; *and others.*

Let $q = a + \alpha$ and $r = b + \beta$; then, since $S\alpha\beta = S\beta\alpha$ and $V\alpha\beta = -V\beta\alpha$,

$\therefore$ $K(qr) = ab + S\alpha\beta - a\beta - b\alpha - V\alpha\beta = (b - \beta)(a - \alpha) = Kr \,.\, Kq$ .....(1).

Again, since rKr and $r + Kr$ are both scalar, therefore

Exp. $= rq + qr + qKr + Kr \,.\, q = q(r + Kr) + (r + Kr)\, q = 4qSr$ .....(2).

9074. (Professor Nilkantha Sarkar, M.A.)—Show that the reciprocal polar of the evolute of the parabola $y^2 = 4ax$, with respect to a circle of radius a, is the cubic $x^3 = y^2(a - 2x)$.

Solution by A. E. Thomas, M.A.; G. G. Storr, M.A.; *and others.*

In *Phil. Trans.* for 1887 ("On a New Form of Tangential Equation," by Prof. Casey) it is shown that, if the tangential equation of a curve is $r = f(\phi)$, the tangential equation of the evolute is $r = f'(\phi) + f(\phi) \cot \phi$; and, moreover, the *polar* equation of its reciprocal is $k^2/\rho = f(\phi) \sin \phi$. Applying these results, we get, for the equation of the evolute of the parabola whose equation is $r = -a \cot^2 \phi$, $r = 2a \cot \phi \operatorname{cosec}^2 \phi - a \cot^3 \phi$. Forming its reciprocal, we get

$$a^2/r = (2a \cot \phi \operatorname{cosec}^2 \phi - a \cot^3 \phi) \sin \phi,$$

$$i.e., \quad a/r = \cos \phi \,(2 + \cot^2 \phi) \quad \text{or} \quad x^3 = y^2(a - 2x).$$

8925. (F. Morley, B.A.)—Prove that, in any acute-angled scalene triangle, $\tan A \tan B \tan C > 3\sqrt{3}$.

Solution by R. F. Davis, M.A.; Prof. Stone, M.A.; *and others.*

In this case, since tan A, tan B, tan C are all positive and unequal, their arithmetical mean is greater than the geometrical mean; or

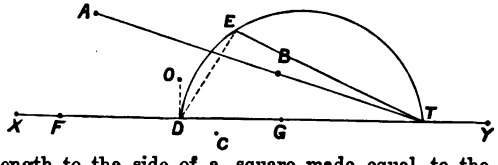
$$\{(\tan A + \tan B + \tan C)/3\}^3 > \tan A \tan B \tan C.$$

But $\tan A + \tan B + \tan C = \tan A \tan B \tan C$; hence $\tan^2 A \tan^2 B \tan^2 C > 27$.

8720. (Professor Edmund Bordage.)—Three points A, B, C and a straight line XY are given in a same plane. Determine the points of intersection of the line XY with the circumference passing through the three points A, B, C, without drawing the circumference.

Solution by C. E. Williams, M.A.; Professor Beyens; *and others.*

Let O be the centre of the circle through A, B, C, and OD perpendicular to XY. Describe a circle on TD as diameter, and in it place a TE



equal in length to the side of a square made equal to the rectangle TA . TB. Cut off DF, DG each equal to DE. F, G are the required points. For $TF \cdot TG = TD^2 - DG^2 = TE^2 = TA \cdot TB$.

8447. (Emile Vigarié.)—Etant donnés deux cercles de centre O et O′, trouvez le lieu des points P tels que sa polaire par rapport au premier cercle soit tangente au second.

Solution by W. J. Barton, M.A.; C. E. Williams, M.A.; *and others.*

Let us choose for coordinate axes any two lines at right angles to each other through the centre O′ of the second circle; then the circle is $x^2 + y^2 = r^2$, and any tangent is $x \cos\phi + y \sin\phi - r = 0$. If the circle O be $x^2 + y^2 + 2gx + 2fy + c = 0$, and (ξ, η) the coordinates of P in any position, the polar is $x(\xi + g) + y(\eta + f) + g\xi + f\eta + c = 0$; and, in order that this may coincide with $x \cos\phi + y \sin\phi - r = 0$, we must have

$$\frac{\cos\phi}{\xi + g} = \frac{\sin\phi}{\eta + f} = \frac{-r}{g\xi + f\eta + c},$$

whence ϕ is at once eliminated, viz., we have

$$\frac{1}{r^2} = \frac{(\xi + g)^2 + (\eta + f)^2}{(g\xi + f\eta + c)^2},$$

which reduces to

$$(g^2 - r^2)\xi^2 + 2fg\xi\eta + (f^2 - r^2)\eta^2 + 2g(c - r^2)\xi + 2f(c - r^2)\eta + c^2 - r^2(f^2 + g^2) = 0,$$

which shows at once that the locus of P is a line of the second order, viz., it is an ellipse, parabola, or hyperbola, according as

$$f^2g^2 <=> (f^2-r^2)(g^2-r^2) \text{ or } f^2+g^2 <=> r^2,$$

the geometric significance of which is $OO' <=>$ radius of O'.

The centre of the conic is the point

$$X = \frac{c-r^2}{r^2-f^2-g^2}g, \quad Y = \frac{C-r^2}{r^2-f^2-g^2}f,$$

which shows that the centre is on the line OO', and for different values of r the locus of the centre is the line $fX = gY$, which is OO'.

The question admits of an easy generalisation, viz., I consider the two conics

$$S \equiv (a, b, c, f, g, h \mathbin{\mathrm{J}} x, y, 1)^2 = 0,$$

$$S' \equiv (a_1, b_1, c_1, f_1, g_1, h_1 \mathbin{\mathrm{J}} x, y, 1)^2 = 0,$$

and, taking P to be (ξ, η), the polar of P with regard to S′ is found to be $\lambda x + \mu y + \nu = 0$, where λ, μ, ν are linear functions of ξ, η, viz., we have

$$\lambda = a_1\xi + h_1\eta + g_1, \quad \mu = h_1\xi + b_1\eta + f_1, \quad \nu = g_1\xi + f_1\eta + c_1,$$

and, the condition that $\lambda x + \mu y + \nu = 0$ may be a tangent to the conic S being

$$(bc-f^2)\lambda^2 + (ca-g^2)\mu^2 + (ab-h^2)\nu^2$$
$$+ 2(gh-af)\mu\nu + 2(hf-bg)\nu\lambda + 2(fg-ch)\lambda\mu = 0,$$

we see immediately, by substituting for λ, μ, ν their values in terms of ξ, η, that the locus of P is a conic; and, this being so when S, S′ are any two conics, it follows that the locus sought cannot be of a degree higher than two when S, S′ degenerate into circles.

It is interesting to note that the locus in question is identical with the envelope of polars of points on S with regard to S′; this is an immediate consequence of the theory of reciprocal polars, but follows with equal ease from direct calculation, viz., (ξ, η) being any point on S, we have

$$a\xi^2 + 2h\xi\eta + b\eta^2 + 2g\xi + 2f\eta + c = 0,$$

and, the polar of (ξ, η) with regard to S′ being $\lambda x + \mu y + \nu = 0$, where λ, μ, ν have the same values as before, the envelope of $\lambda x + \mu y + \nu = 0$, when λ, μ, ν are connected by the quadric relation, is, of course, the same curve as the one found above.

[The locus is obviously the polar reciprocal of the circle O with regard to another circle O′, and is, therefore, a *conic*. See SALMON's *Conic Sections*, Art. 308.]

9125. (R. TUCKER, M.A.)—AD, BE, CF, the altitudes of a triangle, intersect in H, and on them are taken $Ad = HD$, $Be = HE$, $Cf = HF$. Prove that, if λ_1, λ_2, λ_3 are the mid-points of *ef*, *fd*, *de*, then $A\lambda_1$, $B\lambda_2$, $C\lambda_3$ cointersect in the circum-centre; and find the area of *def*.

Solution by R. KNOWLES, B.A., L.C.P.

(1) The trilinear coordinates of *d*, *e*, *f* are respectively proportional to $\cos A$, $\cos B \cos^2 C$, $\cos C \cdot \cos^2 B$; $\cos A \cdot \cos^2 C$, $\cos B$, $\cos C \cdot \cos^2 A$;

$\cos A \cdot \cos^2 B$, $\cos B \cdot \cos^2 A$, $\cos C$; and the equations to $A\lambda_1$, $B\lambda_2$, $C\lambda_3$ are respectively $\beta \cos C - \gamma \cos B = 0$, $\alpha \cos C - \gamma \cos A = 0$, $\alpha \cos B - \beta \cos A = 0$, and these intersect in the centre of the circum-circle.

(2) This I have only been able to obtain in a complicated form.

8821. (Rev. T. C. Simmons, M.A.)—If ABC be a *positively lettered* triangle (*i.e.*, drawn so that, in moving round the triangle from A to B, B to C, C to A, we revolve *positively*), whose *positive* and *negative* Brocard-points are ω, ω', and Lemoine-point K; and if AK, $A\omega$, $A\omega'$ meet BC in X, x, x'; and Y, y, y', Z, z, z' be similarly taken on CA, AB; prove that $x'Y$, Zy are both parallel to BC, and similarly for the corresponding lines. Hence, having given the Lemoine-point of a triangle, show how the Brocard-points may be easily determined.

Solution by Professor Ignacio Beyens.

D'après l'énoncé $\omega AB = \omega BC = \omega CA = \alpha$, et

$$\omega' BA = \omega' AC = \omega' CB = \alpha,$$

(α étant l'angle de Brocard), nous avons

$$\frac{Ax'}{AC} = \frac{\operatorname{sen} ACx'}{\operatorname{sen} Ax'C} = \frac{\operatorname{sen} ACx'}{\operatorname{sen} (B+\alpha)}.$$

Mais $ACx' = 180° - A - Ax'C = 180° - A - (B+\alpha) = 180° - (A+B+\alpha)$,

et par suite
$$\frac{Ax'}{AC} = \frac{\operatorname{sen} (A+B+\alpha)}{\operatorname{sen} (B+\alpha)} = \frac{\operatorname{sen} (C-\alpha)}{\operatorname{sen} (B+\alpha)},$$

et
$$\frac{Bx'}{BC} = \frac{\operatorname{sen} \alpha}{\operatorname{sen} (180° - B - \alpha)} = \frac{\operatorname{sen} \alpha}{\operatorname{sen} (B+\alpha)},$$

et de ces équations on déduira

$$\frac{Ax'}{Bx'} \cdot \frac{BC}{AC} = \frac{\operatorname{sen} (C-\alpha)}{\operatorname{sen} \alpha} = \operatorname{sen} C \cot \alpha - \cos C$$

$$= \frac{2S}{ab} \cdot \frac{a^2+b^2+c^2}{4S} - \frac{a^2+b^2-c^2}{2ab} = \frac{2c^2}{2ab},$$

$$\frac{Ax'}{Bx'} = \frac{c^2}{ab} \cdot \frac{AC}{BC} = \frac{c^2}{ab} \cdot \frac{b}{a} = \frac{c^2}{a^2} = \frac{AY}{CY},$$

donc Yx' est parallèle à BC.

D'après cette propriété pour trouver ω, ω', connaissant le point de Lemoine K, il faut mener BK qui coupe le côté AC en Y, mener Yx' parallèle à BC, joindre Cx', et sur cette droite devra être un des points de Brocard, faire la même construction par rapport à un autre côté et toujours dans le même sens (c'est-à-dire dans les sens ABC ou CBA), et nous aurons de suite les deux points ω, ω'.

8653. (S. Tebay, B.A.)—Trains are dispatched simultaneously at equal intervals from two stations on the same line, in opposite directions,

and with equal speeds. If n trains be dispatched from each station after the first two meet, find the number of conjunctions that have taken place when all are clear.

Solution by the PROPOSER.

Every train from A passes every train from B; there are thus $n \times n = n^2$ conjunctions.

8629. (ASPARAGUS.)—Three normals OP, OQ, OR are drawn from a point O to a parabola whose focus is S; and OP, OQ make complementary angles with the axis; prove that SO is parallel to the tangent at R.

Solution by Rev. T. R. TERRY, M.A.; Rev. H. LONDON, M.A.; *and others.*

If the normal at point am^2, $2am$ pass through point O, whose coordinates are (h, k), we have $k + mh - 2am - am^3 = 0$; therefore

$$m_1 + m_2 + m_3 = 0,\ m_1m_2 + m_2m_3 + m_3m_1 = 2 - h/a,\ m_1m_2m_3 = k/a.$$

By question, $m_1m_2 = 1$; therefore $-m_3^2 = 1 - h/a$, and $m_3 = k/a$; therefore $m_3 = h - a/k$, or tangent at R is parallel to SO. Eliminating m_3, we see that the locus of O is $y^2 = a(x - a)$.

8794. (Professor IGNACIO BEYENS.)—Un bon tirailleur *aux pigeons* a la probabilité de tuer le $66\frac{2}{3}$ *pro cent.* des pigeons; il se propose de tirer 3^{10} dizaines d'oiseaux; lesquels sont les dizaines où il tuera un pigeon, lesquels sont le nombre des dizaines où il tuera deux pigeons, trois, etc?

Solution by the PROPOSER.

Le nombre $\frac{2}{3}$ étant la probabilité de tuer un pigeon; et $\frac{1}{3}$ celle de ne pas tuer à chaque coup; la probabilité de tuer (n) pigeons dans une dizaine et en un certain ordre est $(\frac{2}{3})^n . (\frac{1}{3})^{10-n}$, et la probabilité de tuer le même nombre de pigeons en un ordre quelconque sera

$$\frac{10(10-1)\ldots(10-n+1)}{1.2\ldots n}(\tfrac{2}{3})^n . (\tfrac{1}{3})^{10-n} = \frac{10(10-1)\ldots(10-n+1)}{1.2\ldots n} \cdot \frac{2^n}{3^{10}};$$

c'est-à-dire, que dans 3^{10} dizaines le nombre probable de dizaines où il tuera (n) sera $\frac{10(10-1)\ldots(10-n+1)}{n!} . 2^n$, c'est-à-dire les diverses termes du développement de $(2+1)^{10}$.

8163. (Professor SÂRADÂRANJAN RÂY, M.A. Suggested by Question 8008.)—A conic C cuts two given conics A and B. If two conics be

drawn, one through the common points of A and C and the other through those of B and C; then they cut at four points that lie on the same conic with the common points of A and B.

Solution by EMILY PERRIN, B.Sc.; Professor MATZ, M.A.; *and others.*

Let $A = 0$, $B = 0$, $C = 0$ be the equations of the three conics, and let the conic through the common points of A, C be $A - \lambda C = 0$, and the conic through the common points of B, C be $B - \mu C = 0$; then the four points of intersection of these conics lie on the conic

$$A - \lambda C + \lambda/\mu\,(B - \mu C) = 0,$$

i.e., on $A + \lambda/\mu\, B = 0$, which is a conic through the common points of A, B.

9145. (A. R. JOHNSON, M.A.)—(1) From a point P (x, y) normals PN_1, PN_2, PN_3, PN_4 are drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$; show that

$$\triangle N_2N_3N_4 \,.\, \triangle N_3N_4N_1 \,.\, \triangle N_4N_1N_2 \,.\, \triangle N_1N_2N_3 = a^4b^4c^{-12}\left\{c^4 - a^2x^2 - b^2y^2)^3 - 27a^2b^2c^4x^2y^2\right\}.$$

(2) If two conics cut at the points P_1, P_2, P_3, P_4, show that

$$\triangle P_2P_3P_4 \,.\, \triangle P_3P_4P_1 \,.\, \triangle P_4P_1P_2 \,.\, \triangle P_1P_2P_3 = T/\left\{(\,.\,12\,.\,)^2 - 4\,.\,1^2\,.\,2^2\,.\right\}^3,$$

where T is the tact-invariant (SALMON, p. 325, 5th ed.), and $.\,1^2\,.$, $.\,2^2\,.$ are the discriminants of the two pairs of lines drawn through any point parallel to the asymptotes, and $.\,12\,. = 0$ is the condition that the two pairs should be harmonically conjugate.

Solution by the PROPOSER.

(1) The left-hand member is a rational integral function of the coordinates of P, and vanishes only when P is on the evolute. Moreover, the product is of the 6th degree in the coefficients of P, as is seen by taking P at infinity when two of the points N, say N_3, N_4, move off to infinity.

Then, $\triangle N_1N_3N_4$ and $\triangle N_2N_3N_4$ are each of the order OP^2, and $\triangle N_1N_2N_3$, $\triangle N_1N_2N_4$ are each of the order OP, so that the product is of the order OP^6.

The product varies, therefore, as E, where $E \equiv (c^4 - a^2x^2 - b^2y^2)^3 - 27a^2b^2c^4x^2y^2 = 0$ is the equation to the evolute. Taking P at the centre, the constant comes out as in question.

(2) The product vanishes with T, and becomes infinite when one of the asymptotes of one conic is parallel to one of other conic. The condition that the last should happen is easily seen to be $(ab' + a'b - 2hh')^2 - 4(ab - h^2)(a'b' - h'^2) = 0$, where $ax^2 + 2hxy + by^2$, $a'x^2 + 2h'xy + b'y^2$ are the highest terms of the two conics, or with the notation in question $(\,.\,12\,.)^2 - 4\,.\,1^2\,.\,2^2 = 0$.

Moreover, the product can only contain ratios of the coefficients; it must, therefore, vary as a power of $T/\left\{(\,.\,12\,.)^2 - 4\,.\,1^2\,.\,2^2\,.\right\}^3$; and it

must be the first power, as is seen from (1), E being a tact-invariant of two conics. To determine the constant multiplier take $x^2-1=0$, $y^2-1=0$ for the conics, when there results the expression in question.

8536. (B. HANUMANTA RAU, B.A.)—A, B are two points on the diameter of a circle equidistant from the centre. AP, BQ are any two parallels meeting the circumference. Find the locus of the points of intersection of the tangents at P and Q.

Solution by A. GORDON; G. G. STORR, B.A.; *and others.*

If T be the point of meeting of the tangents, C the centre, $CT=r$, $TCQ=\phi$, $\angle TCB=$ (acute angle) θ; then

$$r=a\sec\phi, \quad 2a\sin\phi=c\sin\theta;$$

therefore $$\frac{a^2}{r^2}+\frac{c^2\sin^2\theta}{4a^2}=1, \quad \text{or} \quad \frac{c^2y^2}{4a^2}+a^2=x^2+y^2.$$

8620. (F. MORLEY, B.A.)—If a transparent visible cube be divided into n^3 equal cubes by transparent visible planes, show that in the figure so formed (1) the number of visible cubes is $\{\frac{1}{2}n(n+1)\}^2$, or is equal to the number of visible rectangles in any face of the figure; (2) the number of visible parallelepipeds (including cubes) is $\{\frac{1}{2}n(n+1)\}^3$. [See Vol. XLIV., p. 109, Quest. 8340.]

Solution by the PROPOSER.

(1) Take the edge of a small cube as unit of length. To get a cube containing r^3 such cubes, we must take pairs of parallel planes at a distance r. From $n+1$ parallel planes we can choose $n-r+1$ pairs at a distance r. Hence there are $(n-r+1)^3$ cubes of edge r. Hence whole number

$$=\sum_{r=1}^{r=n}(n-r+1)^3=\left\{\frac{n(n+1)}{2}\right\}^2.$$

(2) To form the parallelepipeds (boxes would be a preferable term), we take any pair of parallel planes. From $n+1$ planes we get $\frac{1}{2}n(n+1)$ pairs. Hence the whole number is $\{\frac{1}{2}n(n+1)\}^3$.

9105. (A. GORDON.)—Required (1) the tetrahedral coordinates of the centre of the sphere circumscribing the tetrahedron of reference, (2) the conditions amongst the sides that the centre of the circum-sphere may coincide with the centre of gravity.

Solution by ALFRED HARKER, M.A.

1. The equation to the circum-sphere must reduce to the circle

$$a^2\beta\gamma + b^2\gamma\alpha + c^2\alpha\beta = 0,$$

on putting $\delta = 0$, and similarly for the other three sections. Hence the equation to this sphere is

$$a^2\beta\gamma + b^2\gamma\alpha + c^2\alpha\beta + a'^2\alpha\delta + b'^2\beta\delta + c'^2\gamma\delta = 0.$$

If $(\alpha_0\beta_0\gamma_0\delta_0)$ be the centre, its polar plane

$$(c^2\beta_0 + b^2\gamma_0 + a'^2\delta_0)\,\alpha + \ldots + \ldots + \ldots = 0$$

must be identical with $\alpha + \beta + \gamma + \delta = 0$, and hence

$$c^2\beta_0 + b^2\gamma_0 + a'^2\delta_0 = a^2\gamma_0 + c^2\alpha_0 + b'^2\delta_0 = b^2\alpha_0 + a^2\beta_0 + c'^2\delta_0 = a'^2\alpha_0 + b'^2\beta_0 + c'^2\gamma_0.$$

These equations give the ratios of the coordinates.

2. If the centre be also the centre of gravity, $\alpha_0 = \beta_0 = \gamma_0 = \delta_0$, and hence $\quad c^2 + b^2 + a'^2 = a^2 + c^2 + b'^2 = b^2 + a^2 + c'^2 = a'^2 + b'^2 + c'^2$.

These equations reduce to $a = a'$, $b = b'$, $c = c'$.

9023. (N'IMPORTE.)—Les 3 côtés d'un triangle forment une progression arithmétique dont on demande la raison ; on connaît le rapport m de la surface de ce triangle à celle du rectangle construit sur les 2 plus petits côtés, calculer les côtés de ce triangle. Discuter. Dans le cas particulier où $m = \frac{1}{2}$, calculer le rayon du cercle inscrit dans ce triangle.

Solution by W. J. GREENSTREET, M.A. ; G. G. STORR, M.A. ; *and others.*

Taking $2a-b$, $2a$, $2a+b$ as the sides, we have $\Delta = a\,[3\,(a^2-b^2)]^{\frac{1}{2}}$ and $s = 3a$, and the rectangle is $2a\,(2a-b)$. Hence follows the equation for b,

$$b^2\,(4m^2+3) - 16ab^2m + a^2\,(16m^2-3) = 0,$$

whence $b = 8am^2 \pm 3a\,(1-4m^2)^{\frac{1}{2}}$; therefore, if b be real, m is $\not> \frac{1}{2}$.

If $m = \frac{1}{2}$, $b = 2a$, and the Δ is imaginary, $\frac{\Delta}{s}$ becoming ai.

8772. (Rev. T. R. TERRY, M.A.)—If C is the centre of the conic

$$f(x, y) \equiv ax^2 + 2c'xy + by^2 + 2b'x + 2a'y + c = 0,$$

and P is any point (h, k), and if CP, produced if necessary, meets the conic in Q ; show (1) that

$$(ab-c'^2)\,\mathrm{CQ}^2 f(h, k) = (abc + 2a'b'c' - aa'^2 - bb'^2 - cc'^2)(\mathrm{CQ}^2 - \mathrm{CP}^2)\,;$$

and (2) illustrate this result from *à priori* considerations.

Solution by the PROPOSER.

Let $\bar{x}$, $\bar{y}$ be the coordinates of C, and x, y of Q. Then, transforming to C as origin, equation to conic is

$$ax^2 + 2c'xy + by^2 + \mu = 0, \quad \text{where } \mu(ab - c'^2) = abc + \dots,$$

Also
$$\frac{x}{h-\bar{x}} = \frac{y}{k-\bar{y}} = \frac{CQ}{CP};$$

therefore, substituting, $\left(\frac{CQ}{CP}\right)^2 \{f(hk) - \mu\} + \mu = 0$, whence the result.

The *à priori* considerations are:

(1) If P is on the curve $f(h, k) = 0$, therefore CP = CQ, as is obvious.
(2) If curve becomes straight lines $abc + \dots = 0$, therefore $cQ = 0$, as is obvious.
(3) If curve is a parabola $ab - c'^2 = 0$, therefore CP = CQ, as is obvious, each being infinite.

8932. (R. CURTIS, M.A.)—Supposing the discriminant of $S_1 + kS_2$ to be written thus, $k^3\Delta_1 + k^2\Theta_{12} + k\Theta_{21} + \Delta_2$, where

$$S_1 \equiv (a_1b_1c_1f_1g_1h_1)(xyz)^2, \quad S_2 \equiv (a_2b_2c_2f_2g_2h_2)(xyz)^2;$$

T the invariant for S_1, S_2, S_3 $(a_1b_2c_3)^2$ + &c. (see SALMON'S *Conics*, Art. 689, 6th Ed.); and $T' = (A_1B_2C_3)^2$ + &c., the corresponding invariant for the tangential equation; prove that the condition that S_1, S_2, and S_3 should have a common triangle self-conjugate is

$$TT' = \begin{vmatrix} 3\Delta_1 & \Theta_{12} & \Theta_{13} \\ \Theta_{21} & 3\Delta_2 & \Theta_{23} \\ \Theta_{31} & \Theta_{32} & 3\Delta_3 \end{vmatrix}^2.$$

Solution by the PROPOSER.

If the common self-conjugate triangle be taken as the triangle of reference, $S_1 \equiv a_1x^2 + b_1y^2 + c_1z^2$, $S_2 \equiv a_2x^2 + b_2y^2 + c_2z^2$, $S_3 = a_3x^2 + b_3y^2 + c_3z^2$, $A_1 = b_1c_1$, $B_1 = c_1a_1$, $C_1 = a_1b_1$. F_1, G_1, H_1 are each = 0, &c.,

$$\Delta_1 = a_1b_1c_1, \quad \Theta_{12} = a_1A_1 + b_1B_1 + c_1C_1, \text{ &c.},$$
$$T = (a_1b_2c_3)^2, \quad T' = (A_1B_2C_3)^2.$$

But

$$(a_1b_2c_3)(A_1B_2C_3) = \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1, & a_1A_2 + b_1B_2 + c_1C_2, & a_1A_3 + b_1B_3 + c_1C_3 \\ \dots & \dots & \dots \end{vmatrix}$$

$$= \begin{vmatrix} 3\Delta_1 & \Theta_{12} & \Theta_{13} \\ \Theta_{21} & 3\Delta_2 & \Theta_{23} \\ \Theta_{31} & \Theta_{32} & 3\Delta_3 \end{vmatrix}, \text{ therefore, &c.}$$

9109. (R. KNOWLES, B.A.)—Tangents are drawn from a point T (hk) to meet the rectangular hyperbola $xy = a^2$ in P, Q; prove that (1) the equation to the circle TPQ is

$$2hk(x^2 + y^2) - (k^3 + h^2k + 4a^2h)x - (h^3 + hk^2 + 4a^2k)y + 4a^2(h^2 + k^2) = 0;$$

and (2) if the circle meet the hyperbola again in CD, the perpendicular from the centre of the hyperbola on PQ bisects CD, and passes through its pole with respect to the hyperbola.

Solution by ALFRED HARKER, M.A.; SARAH MARKS, B.Sc.; *and others.*

1. The equation to PQ is $kx+hy = 2a^2$, and hence any circle through P, Q can be expressed in the form

$$(kx+hy-2a^2)(hx+ky-\lambda) = (h^2+k^2)(xy-a^2),$$

where $hx+ky=\lambda$ is the other common chord. If (h, k) lie on this circle, we find $2\lambda = h^2+k^2$, and the equation to the circle becomes

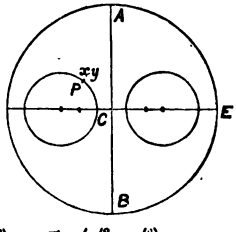
$$2hk(x^2+y^2)-(k^3+h^2k+4a^2h)x-(h^3+hk^2+4a^2k)y+4a^2(h^2+k^2) = 0.$$

2. The equation to CD is then $2hx+2ky = h^2+k^2$, and the diameter $hx = ky$, which is perpendicular to PQ, is thus seen to be conjugate to CD, and therefore bisects it. The same diameter passes through the pole of CD, whose coordinates are $4a^2k/h^2+k^2$ and $4a^2h/h^2+k^2$.

8805. (R. CURTIS, M.A.)—Find (1) the locus of a point in a material lamina, such that if the lamina were moving without rotation in its own plane, and that point were suddenly fixed in space, the ensuing rotation of the lamina would be given; (2) where the point should be if the rotation produced is a maximum; and (3) where the point should be made that the loss of kinetic energy would be given.

Solution by PROPOSER; Professor CHAKRAVARTI, M.A.; *and others.*

1. This locus is a pair of circles. For if C be the C. G. of the lamina, CA the direction of its motion, ω the ensuing angular velocity, V the original velocity, P or xy the point in question, axes of coordinates being CA and CE perpendicular to CA, $\dot{x}\dot{y}$ any other point in the lamina; then $\Sigma m.(x-x').V$ = moment of momentum before P is fixed $= x.M.V$, M = mass of lamina, and



$\Sigma mr^2.\omega$ = moment of momentum after

$$= \Sigma m[(x-x')^2+(y-y')^2]\omega = M(x^2+y^2)\omega+\Sigma m(x'^2+y'^2)\omega;$$

therefore, if k is radius of gyration of lamina round C,

$$(x^2+y^2)\omega+k^2\omega = Vw, \text{ a circle } (a).$$

If ω be of either sign, there will be a pair of circles

$$x^2+y^2+k^2 = \pm\frac{V}{\omega}x.$$

2. This may also be written $\left(x\pm\frac{V}{2\omega}\right)^2+y^2+k^2-\frac{V^2}{4\omega^2} = 0$. Hence it may be seen that, if ω changes in value, the different pairs of circles belong to a coaxal system, and when ω is a maximum the corresponding pair of circles is the pair of limiting points.

3. $\frac{1}{2}MV^2-\frac{1}{2}\Sigma m(x^2+y^2+k^2)\omega^2 = C$, a constant.

But $\omega(x^2+y^2+k^2) = Vx$,

therefore $2Cx^2 = (MV^2-2C)(y^2+k^2)$, a hyperbola.

8922. (Major Hime, R.A.)—If N is the centre of the nine-point circle of the triangle ABC, and if AN produced meets BC in A′, prove that

$$\frac{AN}{NA'} = 1+\frac{2\cos A}{\cos(B-C)}, \quad \frac{BA'}{A'C} = \frac{\sin 2A+\sin 2B}{\sin 2A+\sin 2C}.$$

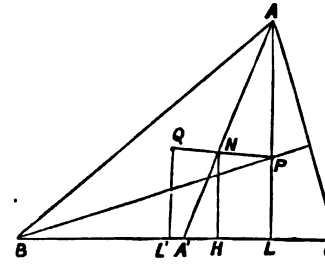
Solution by the Proposer; R. L. Knowles, B.A.; *and others.*

The trilinear coordinates of P, Q, N are

$(R\cos A, R\cos B, R\cos C)$,

$(2R\cos B\cos C, 2R\cos C\cos A, 2R\cos A\cos B)$,

$\left[\tfrac{1}{2}R\cos(B-C), \tfrac{1}{2}R\cos(C-A), \tfrac{1}{2}R\cos(A-B)\right]$;



hence we have

$$\frac{AN}{NA'} = \frac{AL-NH}{NH}$$

$$= \frac{2R\sin B\sin C-\tfrac{1}{2}\cos(B-C)}{\tfrac{1}{2}\cos(B-C)}$$

$$= \frac{4\sin B\sin C-\cos(B-C)}{\cos(B-C)} = 1+\frac{2\cos A}{\cos(B-C)}.$$

$$\frac{BA'}{A'C} = \frac{\sin C}{\sin B}\,\frac{\cos(A-B)}{\cos(C-A)} = \frac{\sin 2A+\sin 2B}{\sin 2A+\sin 2C}.$$

8988. (Professor Ignacio Beyens.)—Si a, b, c sont les côtés d'un triangle quelconque $(a<b<c)$, a_1, a_2 les bissectrices de l'angle A et son supplémentaire et l'on construit un triangle avec les côtés a_1, a_2 et ayant A pour angle compris; et de la même manière l'on fait avec les bissectrices β_1, β_2 des angles B, γ_1, γ_2 des angles C, et l'on appelle S_1, S_2, S_3 les surfaces de ces triangles, on aura $S_1^{-1}+S_3^{-1} = S_2^{-1}$.

Solution by C. E. Williams, M.A.; Hannah Noylan; *and others.*

$$a_1 = \frac{b\sin C}{\sin(B+\tfrac{1}{2}A)}, \quad a_2 = \frac{b\sin C}{\cos(B+\tfrac{1}{2}A)},$$

whence $S_1 = \dfrac{b^2\sin^2 C\sin A}{\sin(2B+A)} = \dfrac{b^2\sin^2 C\sin A}{\sin(C-B)} = \dfrac{abc\sin A\sin B\sin C}{a\sin(C-B)}$;

therefore, as $a\sin(C-B)+b\sin(A-C)+c\sin(B-A) = 0$,

it follows that $S_1^{-1}+S_3^{-1} = S_2^{-1}$.

9198. (Professor CROFTON, F.R.S.)—Prove that

$$e^{\frac{1}{2}a(x+D)^2}\, e^{\frac{1}{2}bx^2} = (1-a-ab)^{-\frac{1}{2}}\, e^{\frac{1}{2}[(a+b+ab)/(1-a-ab)]x^2}.$$

Solution by Professor SIRCOM, M.A.; J. W. SHARPE, M.A.; *and others.*

We have $$f(x+D)\,u = e^{-\frac{1}{2}x^2} f(D)\, e^{\frac{1}{2}x^2}\, u.$$

Hence $$e^{\frac{1}{2}a(x+D)^2}\, e^{\frac{1}{2}bx^2} = e^{-\frac{1}{2}x^2}\, e^{\frac{1}{2}aD^2}\, e^{\frac{1}{2}(1+b)x^2}.$$

But $$e^{hD^2}\, e^{-kx^2} = (4hk+1)^{-\frac{1}{2}}\, e^{-k/(4hk+1)\,x^2},$$ as is well known,

whence, putting $2h = a$, $-2k = 1+b$, and simplifying, we obtain the required result.

8864 & 9004. (Professor SYLVESTER, F.R.S.) — (8864.) Extend BEDETTI's theorem concerning the intersection of a surface with a tangent plane, by showing that the intersection of a hyper-surface with its linear and quadratic polars in respect to any given point upon it, is a curve of double curvature having a sextuple point at the given point.

9004. (W. J. C. SHARP, M.A.)—A hyper-surface, in space of n dimensions, is intersected by the linear polar of any point upon it in a hyper-surface (in space of $n-1$ dimensions) having a node at the given point. If the hyper-surface be a quadric (in space of four dimensions) the intersection is a quadric cone whose vertex is at the given point.

Solution by W. J. C. SHARP, M.A.

In a paper read before the *London Mathematical Society*, in December 1883, I have shown that, $(\lambda, \mu \ldots \rho)$ being $p+1$ in number, if

$$\frac{\lambda x_1 + \mu x_2 + \ldots + \rho x_{p+1}}{\lambda+\mu+\ldots+\rho}, \quad \frac{\lambda y_1 + \mu y_2 + \ldots + \rho y_{p+1}}{\lambda+\mu+\ldots+\rho}, \text{ \&c.}$$

be substituted for x, y, &c., respectively (the coordinates of a point in space of m dimensions), in $U = 0$, the equation to a locus of the n^{th} order in that space (m being $\not< p$), and if $(x_1, y_1 \ldots)$, $(x_2, y_2 \ldots)$, &c. be the vertices of a simplicissimum (Quest. 8242) of the p^{th} order, the resulting equation in $\lambda, \mu \ldots \rho$ will be the equation to the section of the given locus ($U = 0$) by the linear locus (in space of p dimensions) determined by the $p+1$ vertices (with the above as simplicissimum of reference).

If $m = p+1$ and U_1 be the value of U, when $(x_1, y_1 \ldots)$ are written for $(x, y \ldots)$, and if Δ_r stand for $\left(x_r \frac{d}{dx_1} + y_r \frac{d}{dy_1} + \ldots\right)$, the resulting equation is

$$\lambda^n U_1 + \lambda^{n-1}(\mu\Delta_2 + \nu\Delta_3 + \ldots)\, U_1 + \frac{1}{1.2}\lambda^{n-2}(\mu\Delta_2 + \nu\Delta_3 + \ldots)\, U_1 + \text{\&c.} = 0,$$

and if $(x_1, y_1 \ldots)$ lie upon $U = 0$, and the linear locus of section be the linear polar of that point, $\Delta_2 U_1$, $\Delta_3 U_1$, &c. vanish, and the locus of section has a node at $(x_1, y_1 \ldots)$, and if $m = 4$ and $n = 2$ this is a quadratic cone with vertex at that point (9024).

If the points $(x_1, y_1 \ldots)$, $(x_2, y_2 \ldots)$, &c. also lie on the quadratic polar $(\mu\Delta_2 + \nu\Delta_3 + \ldots)^2 U_1 = 0$, the common locus is determined by this equation, and

$$\frac{1}{1.2.3}(\mu\Delta_2 + \nu\Delta_3 + \ldots)^3 U_1 + \frac{1}{1.2.3.4}(\mu\Delta_2 + \nu\Delta_3 + \ldots)^4 U_1 + \&c. = 0,$$

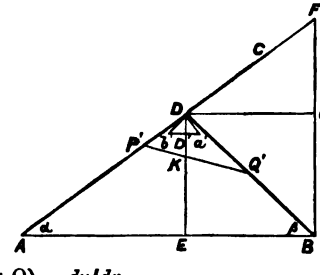
and therefore has a sextuple point at the given point. If $m = 4$, this locus is a curve of double curvature.

In the same way, in space of m dimensions, the intersection of hypersurface with its linear, quadratic, ... $(m-2)$-ic polars, with respect to a point on it, is a curve of double curvature, having a point of order $1.2.(m-1)$ at the given point.

9228. (Rev. T. P. KIRKMAN, F.R.S.)—If two uniform rods AC, BD, whose ends A, B rest on a smooth horizontal plane, are connected by a hinge at D, and tied together with a string AB, find the tension of AB.

Solution by D. BIDDLE; J. W. SHARPE, M.A.; *and others.*

Since action is equal to reaction, the required tension bears to the combined weight of the rods a ratio which is inverse to that which an infinitesimal shortening of the string bears to the consequent elevation of the centre of gravity of the arrangement.



et P = weight of AC,

Q = weight of BD,

$\alpha = \angle CAB$, $\beta = \angle DBA$,

x = length of string, y = height of centre of gravity, K. Then $T / (P + Q) = dy/dx$.

The ratio of the shortening of the string to the elevation of D (the apex of the triangle) is $1 : 1 / (\tan\alpha + \tan\beta)$. For, let $DE = u$, $AE = w$; then $w = (AD^2 - u^2)^{\frac{1}{2}}$, and $dw / du = -u / (AD^2 - u^2)^{\frac{1}{2}} = -\tan\alpha$. Similarly, $-\tan\beta$ for the other side. So that AB shortens $(\tan\alpha + \tan\beta)$ whilst DE lengthens by unity. But $y = \frac{1}{2}\,DE\left\{\frac{AC - AD}{AD} \cdot \frac{P}{P+Q} + 1\right\}$,

therefore
$$\frac{dy}{dx} = \frac{1}{\tan\alpha + \tan\beta} \cdot \left\{\frac{P.AC + Q.AD}{2(P+Q)\,AD}\right\},$$

and
$$T = \frac{P.AC + Q.AD}{2AD\,(\tan\alpha + \tan\beta)},$$

which agrees with the formula (WALTON's *Mech. Prob.* 1855, p. 105),

$$T = \frac{P.AC.\sin\alpha + Q.BD.\sin\beta}{2AB.\tan\alpha\tan\beta},$$

found by taking moments about A, D, and B, for

$$AB \cdot \tan\alpha = DE(1 + \tan\alpha \cot\beta) = DE(\tan\alpha + \tan\beta \,/\, \tan\beta,$$

and $$P \cdot AC \cdot \sin\alpha + Q \cdot BD \cdot \sin\beta = (P \cdot AC + Q \cdot AD)\, DE \,/\, AD.$$

At the same time, the present formula is simpler.

9113. (Professor HAUGHTON, F.R.S.)—The most recently found value of the Sun's distance depends upon Prof. Newcomb's determination of the velocity of light. This velocity Professor Newcomb values at 299,860 kilometres per second, with a probable error of 30 kilometres either way. Combining this with Nyrén's value of the constant of aberration 20″·492, you are required to find the solar parallax and the distance of the Sun in kilometres and British miles.

Solution by Professors DE WACHTER, BEYENS, *and others.*

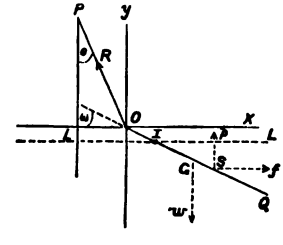
The Sun's distance calculated upon these data

$$x = \frac{31558150 \cdot 20{\cdot}492 \cdot 299860}{1296000} = 149600000 \text{ kilom.}$$

Hence the Sun's parallax = 8″·78.

9180. (CHARLOTTE A. SCOTT, B.Sc.)—One end of a string is fixed above a uniformly flowing stream; the other end is attached to the middle point of the end of a plank which is partially immersed in the water, with its plane opposed to the stream: obtain equations for determining the position of equilibrium, and show that the string and plank cannot be in the same straight line.

Solution by Professor F. X. DE WACHTER.



Let PO = r be the string, OQ = a = the plank weighing w, and having a specific gravity = δ. Let p denote the intensity of the upward pressure, f that of the uniform stream, θ the angle between OP and the vertical PL, ω that between OQ and PL, PL = h = distance of P from the stream-level, and R the tension of the string. The plank being (very nearly) homogeneous, rectangular, and partially immersed, we are led to suppose G, point of application of w, in the middle of OQ, and the points of application of f and p to meet in S, the middle point of the immersed part IQ.

Draw a vertical and a horizontal axis through O, and apply all the forces there. The conditions of equilibrium are

$$\tfrac{1}{2}wa\sin\omega - \mathrm{OS}\,(f\cos\omega + p\sin\omega) = 0 \quad \ldots\ldots\ldots\ldots (1),$$

$$p - w + \mathrm{R}\cos\theta = 0, \quad f - \mathrm{R}\sin\theta = 0 \quad \ldots\ldots\ldots\ldots (2, 3).$$

The values of f and p, and the distance OS, depend on the immersed part IQ. To the above equations we have therefore to add (k being a constant)

$$f = \frac{k\,(a\cos\omega + r\cos\theta - h)}{\cos\omega}, \quad p = \frac{w\,(a\cos\omega + r\cos\theta - h)}{\delta a\cos\omega} \quad \ldots (4, 5).$$

$$\mathrm{OS} = \frac{a\cos\omega - r\cos\theta + h}{2\cos\omega} \quad \ldots\ldots\ldots\ldots (6).$$

These six equations determine the values of ω, θ, f, p, OS, and R.

Finally, if POQ were a straight line, then the resultant of p and f should pass through G, *i.e.*, S ought to coincide with G. The plank being partially immersed and homogeneous, this is impossible, and therefore, &c.

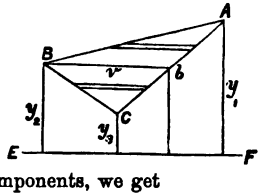
8922. (W. S. B. Woolhouse.) Any straight line passing through the centroid being taken as an axis in a thin lamina having a quadrilateral boundary, let y_1, y_2, y_3, y_4, with relative algebraic signs, denote perpendiculars drawn thereon from the four corners; then, if k be the radius of gyration of the lamina round the said axis, show that

$$6k^2 = -y_1y_2 - y_1y_3 - y_1y_4 - y_2y_3 - y_2y_4 - y_3y_4.$$

[It is also remarkable that the same formula holds good when the axis passes through the centre of the four corners.]

Solution by the Proposer.

Preparatory to an investigation of this question, it is proposed first to consider the more simple case of a triangular lamina ABC, the mass or comparative area of the lamina being denoted by m, and the three perpendiculars upon the axis as stated in the diagram.



In the diagram the line EF is taken as the axis of rotation; and Bb, drawn parallel thereto, divides the lamina into two component triangular portions, ABb, bBC.

Adding the respective moments of these components, we get

$$mk^2 = \int_{y_2}^{y_1} y^2\,\frac{v\,(y_1 - y)}{y_1 - y_2}\,dy + \int_{y_3}^{y_2} y^2\,\frac{v\,(y - y_3)}{y_2 - y_3}\,dy = v\,\frac{y_1\frac{1}{3}y^3 - \frac{1}{4}y^4}{y_1 - y_2} + v\,\frac{\frac{1}{4}y^4 - y_3\frac{1}{3}y^3}{y_2 - y_3}$$

= (between respective limits)

$$v\left(\frac{y_1}{3}\,\frac{y_1^3 - y_2^3}{y_1 - y_2} - \frac{1}{4}\,\frac{y_1^4 - y_2^4}{y_1 - y_2}\right) + v\left(\frac{1}{4}\,\frac{y_2^4 - y_3^4}{y_2 - y_3} - \frac{y_3}{3}\,\frac{y_2^3 - y_3^3}{y_2 - y_3}\right)$$

$$= v\left[y_1\tfrac{1}{3}(y_1^2 + y_1y_2 + y_2^2) - \tfrac{1}{4}(y_1^3 + y_1^2y_2 + y_1y_2^2 + y_2^3)\right]$$
$$+ v\left[\tfrac{1}{4}(y_2^3 + y_2^2y_3 + y_2y_3^2 + y_3^3) - y_3\tfrac{1}{3}(y_2^2 + y_2y_3 + y_3^2)\right]$$

$$= v\tfrac{1}{12}\left[y_1^3 - y_3^3 + y_2(y_1^2 - y_3^2) + y_2^2(y_1 - y_3)\right]$$

$$= \tfrac{1}{12}v\,(y_1 - y_3)\,(y_1^2 + y_1y_3 + y_3^2 + y_1y_2 + y_2y_3 + y_2^2).$$

But $m = \frac{1}{2}v(y_1 - y_2 + y_2 - y_3) = \frac{1}{2}v(y_1 - y_3)$;

therefore $6k^2 = y_1^2 + y_2^2 + y_3^2 + y_1y_2 + y_1y_3 + y_2y_3$ .....(A).

If the axis pass through the centre of gravity of the triangle,

$$y_1 + y_2 + y_3 = 0, \quad \text{and} \quad 0 = y_1^2 + y_2^2 + y_3^2 + 2y_1y_2 + 2y_1y_3 + 2y_2y_3,$$

therefore $6k^2 = -y_1y_2 - y_1y_3 - y_2y_3$.....(B),

which is analogous to the formula in the question.

If the axis of revolution pass through C, then $y_3 = 0$, and

$$6k^2 = y_1^2 + y_1y_2 + y_2^2 \text{.....(C)}.$$

We now proceed to apply the results for the triangle to the more general case of the quadrilateral. Suppose the axis of revolution to pass through any assigned point within the quadrilateral, and conceive lines to be drawn from this point to the four corners, thus dividing the quadrilateral into four component triangles; also let x_1, x_2, x_3, x_4 denote the abscissæ measured on the axis. Then, by applying the formula (C) to obtain the moments of these component triangles, and summing the four results, we get

$$\begin{aligned}
12mk^2 &= \Sigma\, 2\Delta\,(y_1^2 + y_2^2 + y_1y_2) \\
&= (x_1y_2 - x_2y_1)(y_1^2 + y_2^2 + y_1y_2) \ldots + (x_4y_1 - x_1y_4)(y_4^2 + y_1^2 + y_1y_4) \\
&= \Sigma x_1(y_1^2y_2 + y_2^3 + y_1y_2^2 - y_4^3 - y_1^2y_4 - y_1y_4^2) \\
&= \Sigma x_1(y_2 - y_4)(y_1^2 + y_1y_2 + y_1y_4 + y_2^2 + y_2y_4 + y_4^2) \\
&= \Sigma x_1(y_2 - y_4)\{(y_1 + y_2 + y_4)^2 - y_1y_2 - y_1y_4 - y_2y_4\} \text{ .....(}\alpha\text{)},
\end{aligned}$$

in each of which expressions, under the summation symbol Σ, the sub-numerals have to pass through the cycle 1234 to include the remaining terms.

In order that the origin may be at the centre of the area of the quadrilateral, we must have the condition

$$\begin{aligned}
0 &= \Sigma 2\Delta\,(y_1 + y_2) = (x_1y_2 - x_2y_1)(y_1 + y_2) \ldots + (x_4y_1 - x_1y_4)(y_4 + y_1) \\
&= \Sigma x_1(y_1y_2 + y_2^2 - y_4^2 - y_1y_4) = \Sigma x_1(y_2 - y_4)(y_1 + y_2 + y_4).
\end{aligned}$$

Multiply this by $y_1 + y_2 + y_3 + y_4$, which, as regards the cycle, may be treated as a constant, and it becomes

$$0 = \Sigma x_1(y_2 - y_4)\{(y_1 + y_2 + y_4)^2 + y_3(y_1 + y_2 + y_4)\} \text{ .....(}\beta\text{)}.$$

Subtract (β) from (α), and

$$\begin{aligned}
12mk^2 &= \Sigma x_1(y_2 - y_4)\{-y_1y_2 - y_1y_4 - y_2y_4 - y_3(y_1 + y_2 + y_4)\} \\
&= \{-y_1y_2 - y_1y_3 - y_1y_4 - y_2y_3 - y_2y_4 - y_3y_4\}\,\Sigma x_1(y_2 - y_4),
\end{aligned}$$

because the factor within the brackets is symmetrical, and may also be treated as a constant.

But $2m = 2\Sigma\Delta = (x_1y_2 - x_2y_1) \ldots + (x_4y_1 - x_1y_4) = \Sigma x_1(y_2 - y_4)$.

Therefore, by division, $6k^2 = -y_1y_2 - y_1y_3 - y_1y_4 - y_2y_3 - y_2y_4 - y_3y_4 \cdots (\gamma)$,

which is the formula stated in the question.

The expression (A) for the triangle may be written

$$12k^2 = y_1^2 + y_2^2 + y_3^2 + (y_1 + y_2 + y_3)^2.$$

Distinguishing by k_1, k_2, k_3, k_4 the values for the triangles ABC, BCD, CDA, DAB, respectively, we have

$$12k_1^2 = y_1^2+y_2^2+y_3^2+(y_1+y_2+y_3)^2 \quad | \quad 12k_3^2 = y_3^2+y_4^2+y_1^2+(y_3+y_4+y_1)^2,$$
$$12k_2^2 = y_2^2+y_3^2+y_4^2+(y_2+y_3+y_4)^2 \quad | \quad 12k_4^2 = y_4^2+y_1^2+y_2^2+(y_4+y_1+y_2)^2.$$

If the axis of revolution pass through the centre of the four points A, B, C, D, then $y_1+y_2+y_3+y_4 = 0$, and therefore, in this case,

$$6k_1^2 = 6k_2^2 = 6k_3^2 = 6k_4^2 = \tfrac{1}{2}(y_1^2+y_2^2+y_3^2+y_4^2) \quad \ldots\ldots\ldots (1),$$
$$= -y_1y_2-y_1y_3-y_1y_4-y_2y_3-y_2y_4-y_3y_4 \quad \ldots\ldots\ldots\ldots (2),$$

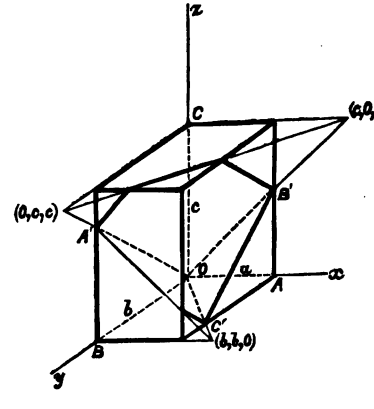
because $(1)-(2) = \tfrac{1}{2}(y_1+y_2+y_3+y_4)^2 = 0.$

The expression (2), being identical with (γ), is therefore applicable whether the axis passes through the centre of the *points* A, B, C, D, or through the centre of the *area* ABCD, which is somewhat remarkable. But the formula (1) will not apply generally to the latter case, because then $y_1+y_2+y_3+y_4$ does not necessarily vanish.

9015. (Rev. T. C. Simmons, M.A.)—The lengths of three lines are known not to exceed a, b, c, respectively; find the chance of their forming a possible triangle.

Solution by Professor Schoute.

Suppose $a<b<c$, and consider the three lines x, y, z as the coordinates of a point in space. Then the number of possible cases is measured by the volume of the rectangular parallelopipedon OABC, that of the favourable cases by the volume of the portion of this parallelopipedon included within the trihedron O(A'B'C') formed by the planes $x = y+z$, $y = z+x$, $z = x+y$, the edges of which are situated in the planes YOZ, ZOX, XOY. Now the portion of the parallelopipedon P situated under the plane $x = y+z$ is a tetrahedron with three conterminous edges equal to a and rectangular by two; we represent it by T(a).



As the figure shows, the part of P under the plane $y = z+x$ is the difference between two such tetrahedra, and can be represented by T(b) − T($b-a$). In the same way, the part of P behind the plane $z = x+y$ can be repre-

sented by $T(c)-T(c-a)-T(c-b)$. The required chance is therefore

$$W = \frac{P-T(a)-T(b)-T(c)+T(b-a)+T(c-a)+T(c-b)}{P},$$

when the symbols P and T represent also the volume of the solids in question. As $P = abc$ and $T(p) = \frac{1}{6}p^3$, one finds

$$W = \frac{6abc-a^3-b^3-c^3+(b-a)^3+(c-a)^3+(c-b)^3}{6abc},$$

and when a, b, c are equal $W = \frac{1}{4}$.

9024 & 9071. (Professor SYLVESTER, F.R.S.)—(9024.) For greater distinctness, the name of *Hyper-cartesian* (not to be confounded with a *hyper-cartesic*) being given to that particular form of the bicircular quartic in which four concyclic foci become collinear; prove that, if four points are given in a plane, the locus of the curve *in space* whose distances from any three of them are subject to a given homogeneous linear relation is a curve of the 4th order. (This spaced curve may be termed a Hyper-cartesic.)

9071. If A, B, C, D be the four points mentioned in Quest. 9024, F, G, H, K four other points in which the hyper-cartesic cuts the plane passing through the first named four, a bicircular quartic may be drawn through A, B, C, D, of which F, G, H, K will be concyclic foci.

Solution by A. R. JOHNSON, M.A.

Let the distances from A, B, C, D be denoted by r_1, r_2, r_3, r_4. Then we have, at every point of the locus,

$$l_{12}r_2+l_{13}r_3+l_{14}r_4 = 0;\quad l_{21}r_1+l_{23}r_3+l_{24}r_4 = 0;\quad l_{31}r_1+l_{32}r_2+l_{34}r_4 = 0;$$
$$l_{41}r_1+l_{42}r_2+l_{43}r_3 = 0;$$

demanding the vanishing of the minors of the determinant

$$\begin{vmatrix} 0, & l_{12}, & l_{13}, & l_{14} \\ l_{21}, & 0, & l_{23}, & l_{24} \\ l_{31}, & l_{32}, & 0, & l_{34} \\ l_{41}, & l_{42}, & l_{43}, & 0 \end{vmatrix}.$$

From the first two relations we have

$$\frac{1}{l_{13}l_{14}}(l_{13}^2r_3^2+l_{14}^2r_4^2-l_{12}^2r_2^2) = \frac{1}{l_{23}l_{24}}(l_{23}^2r_3^2+l_{24}^2r_4^2-l_{21}^2r_1^2)\ (=-2r_3r_4).$$

The curve is therefore on a sphere and the intersection of

$$l_{12}r_2+l_{13}r_3+l_{14}r_4 = 0$$

by any plane is a bicircular quartic. Hence the points where the curve cuts the plane are the intersections of a circle and a bicircular quartic; and, as the circular points at infinity in the plane each count for two, there are only $8-4 = 4$ other points of intersection. Thus the curve is of the fourth order.

Let r', r'', &c. denote distances from F, G, &c. Then, substituting the coordinates of F, G, H, K in the first relation, and eliminating the l's, we get

$$\begin{Vmatrix} r'_1, & r''_1, & r'''_1, & r''''_1 \\ r'_2, & r''_2, & r'''_2, & r''''_2 \\ r'_3, & r''_3, & r'''_3, & r''''_3 \end{Vmatrix} = 0.$$

And F, G, H, K have been proved concyclic (because the curve is spherical), and the distances of A, B, C from them are connected by linear relations; therefore ABC, and in the same way BCD, &c., lie on a bicircular quartic having F, G, H, K as concyclic foci. Therefore all four points A, B, C, D do so.

[Another solution of Quest. 9024 is given on p. 160 of Vol. XLVII.]

9171. (H. L. ORCHARD, B.Sc., M.A.)—Show, by an ordinary quadratic method, that the roots of the equation

$$u^4 + 11u^3 + 44u^2 + 132u + 144 = 0$$

are $u = \frac{1}{2}\{[22(41)^{\frac{1}{2}} - 30]^{\frac{1}{2}} - [(41)^{\frac{1}{2}} + 11]\} = -1{\cdot}71843.$

Solution by D. BIDDLE; Professor IGNACIO BEYENS; *and others.*

By reduction, we obtain $(u^2+12)^2 + 11u(u^2+12) + 20u^2 = 0$, whence $u^2 + 12 = \frac{1}{2}[\pm(41)^{\frac{1}{2}} - 11]u$, and the stated result follows.

4129. (The EDITOR.)—If three given weights (considered as heavy material points) are attached to the surface of a sphere, find (1) the position of equilibrium of the sphere when resting on a horizontal plane; and (2) give also the result in the particular case in which the weights are arranged in a great circle.

Solution by D. BIDDLE.

Let the sphere have marked upon it two great circles at right angles to each other, by reference to which the latitude and longitude of the weights may be defined. Let A, B, C be the weights; A_1, B_1, C_1 their respective latitudes; and A_2, B_2, C_2 their respective longitudes.

To determine their relative distances, or the sides of the plane triangle of which they form the apices, we first draw perpendiculars from each to the equatorial plane. The cosines of A_1, B_1, C_1 will give the respective distances of the points so found, from the centre of the sphere. Then $B_2 - A_2$ will be the angle separating the "meridians" of A and B; and, similarly, $C_2 - B_2$ of B and C. From these data (two sides and included angle in each case) we can find the distances between the several perpendiculars. The distances thus found each form one side of a right-angled triangle, a second side of which is the difference (or it may be the sum) of the sines of the latitudes of two of the weights, and the third side or

hypotenuse is the actual distance between those two weights. The three hypotenuses having been found, we easily construct a triangle having them for its several sides, and circumscribe a circle about it. This corresponds with the circle connecting the three weights.

In the following diagram (Fig. 1) let A, B, C, the three weights, be as 1, 2, 3; then their centre of gravity is found as follows:—

$$AD = \frac{B.BD}{A} = \frac{B.AB}{A+B} = \tfrac{2}{3}AB.$$

Similarly,

$$CE = \frac{(A+B)\,DE}{C} = \frac{(A+B)\,CD}{A+B+C} = \tfrac{1}{2}CD.$$

Therefore E, the centre of gravity, in this case bisects CD. When the sphere is in equilibrium, with these weights upon it, the centre of gravity, E, must be directly under the centre of the sphere (unless identical with it); and if the line joining them be produced, it will indicate the point on the surface at which the sphere will touch the horizontal plane whereon it rests.

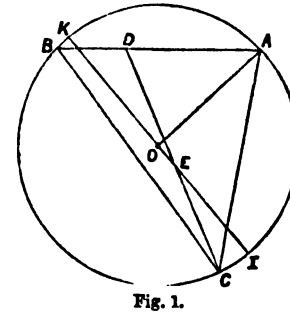


Fig. 1.

To find the position of this point, we must first find the position of the extremity of that radius of the sphere which passes through O, the centre of the circle ABC. Let GIKH (Fig. 2) be a portion of the plane of a great circle, containing O, the centre of the smaller circle ABC, and E, the centre of gravity of the weights, and let F be the centre of the sphere. Draw the radius FL through O, and FP through E. Then IK, the diameter of the smaller circle ABC, having already been found, FO, OL, LK easily follow.

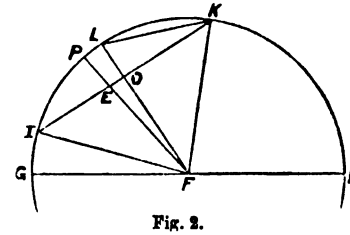


Fig. 2.

Then, on the surface of the sphere, with radius LK, describe arcs of circles (as bases of cones) from A, B, C (as apices of the same). Their point of intersection will be L, the extremity of the diameter of the sphere passing through O. From L, with the same radius LK, the circle ABC can now be drawn on the sphere. Then, OE having been found from data represented in Fig. 1, the positions of I and K, as related to A, B, C, can also be determined; and, if an arc of a great circle be drawn from I to K through L, P will be a point in this arc at a distance from L easily determined, since FO, OE, and FL are given.

When the weights are arranged in a great circle, as in the particular case referred to in the question, P must be on the same great circle (*e.g.*, at I in Fig. 1), unless E coincide with the centre of the sphere, when P may occupy any point whatever on the surface, or rather *none*, because there will be no stable equilibrium.

8755. (Professor Neuberg.)—On prolonge les hauteurs du triangle ABC au-delà des sommets des quantités $AA' = BC$, $BB' = CA$, $CC' = AB$; démontrer : (1) les triangles ABC, A'B'C' ont le même centre de gravité ; (2) si α, α' sont les angles de Brocard de ces triangles, on a

$$A'B'C' = 2ABC\,(2 + \cot\alpha),\quad (A'B')^2 + (B'C')^2 + (A'C')^2 = 4ABC\,(3 + 2\cot\alpha),$$

$$\cot\alpha' = \frac{2\cot\alpha + 3}{2\cot\alpha + 4};$$

(3) les points A, B, C sont les centres des carrés construits, intérieurement, sur les côtés du triangle A'B'C' ; (4) les milieux des côtés de A'B'C' sont les centres des carrés construits, extérieurement, sur les côtés de ABC.

Solution by Professor Ignacio Beyens, M.A.

(3) D'abord les triangles ACC', ABB' étant égaux, nous avons

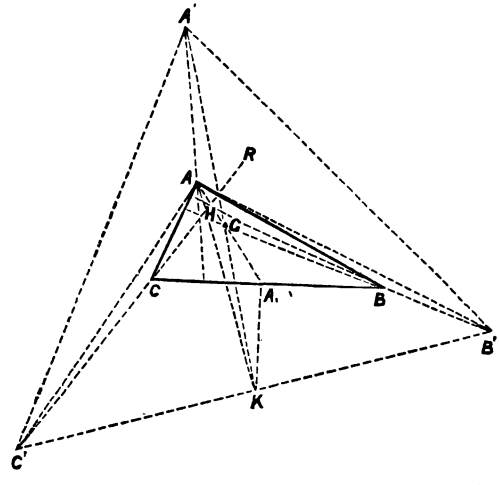
$$AC' = AB',\ \text{angle } CAC' = BB'A,\ HC'B' + HB'C' = 180° - C'HB' = A,$$

$$CC'A + CAC' = ACH = 90° - A;$$

donc $$(HC'B' + CC'A) + (HB'C' + CAC') = 90°,$$

ou $$AC'B' + AB'C' = 90°,\ 2AC'B' = 90°,\ AB'C' = AC'B' = 45°;$$

donc A est le centre du carré construit sur B'C' ; de même B, C sont les centres des carrés analogues ; ce qui est la troisième propriété.



(4) Soit K le milieu de B'C', les triangles AKB, CKC' sont égaux, parce que $CC' = AB$ par hypothèse, $AK = KC'$ parce que $AC'B' = 45° = C'AK$ d'après ce qui précède, et $AC'K = KAB'$, $AC'C = BAB'$, et soustrayant on a $CC'K = KAB$, donc $CK = KB$.

D'ailleurs, nous avons $CKC' = AKB$, $BKB' = AKC$, et par suite

$$CKC' + BKB' = AKB + AKC = CKB,$$

ou $\quad 180° - CKB = CKB$; d'où $2CKB = 90°$, $CKB = 45°$,

donc K est le centre du carré construit sur BC, et de même manière les milieux de A'C', B'A' seront les centres des carrés construits sur AC, BA, qui est la 4ème partie.

(1) Soit A_1 le milieu de BC; joignons AA_1, A'K, qui se coupent en G; les triangles semblables GAA', GA_1K nous donnent

$$\frac{GA_1}{GA} = \frac{A_1K}{AA'} = \frac{\frac{1}{2}BC}{BC} = \frac{1}{2}, \quad GA_1 = \frac{1}{2}GA, \quad \frac{GK}{GA'} = \frac{1}{2}, \quad GK = \frac{1}{2}GA',$$

c'est à dire, que G est le centre de gravité des deux triangles ABC, A'B'C', qui est la 1ère partie.

(2) Pour démontrer les autres parties de la question, nous aurons

$$C'B'^2 = 2AC'^2 = 2(b^2 + c^2 - 2bc\cos ACC') = 2(b^2 + c^2 + 2bc\sin A)$$

dans le triangle ACC'... ; de même

$$A'B'^2 = 2(a^2 + b^2 + 2ab\sin C), \quad A'C'^2 = 2(a^2 + c^2 + 2ac\sin B),$$

mais $ab\sin C = bc\sin A = ac\sin B = 2S$; donc

$$A'B'^2 + B'C'^2 + A'C'^2 = 2(2a^2 + 2b^2 + 2c^2 + 12S),$$

$$A'B'^2 + B'C' + A'C'^2 = 4(a^2 + b^2 + c^2 + 3S),$$

$$A'B'^2 + B'C'^2 + A'C'^2 = 4S(2\cot\alpha + 3),$$

$$\frac{a^2 + b^2 + c^2}{2S} = \cot\alpha, \quad a^2 + b^2 + c^2 = 2S \cdot \cot\alpha,$$

qui est la formule (2) de l'énoncé.

Soit O le centre du cercle circonscrit au triangle donné; nous aurons

$$HA = 2OA_1 = 2CA_1\cot A = a\cot A,$$

et par suite, $\quad HA' = AA' + HA = a(1 + \cot A),$

$$HB' = b(1 + \cot B), \quad HC' = c(1 + \cot C),$$

et $$\frac{A'HB'}{ABC} = \frac{HA' \cdot HB'}{AC \cdot BC} = \frac{ab(1 + \cot A)(1 + \cot B)}{ab} = (1 + \cot A)(1 + \cot B)$$

(les triangles A'HB', ABC ayant les angles A'HB', ACB, qui sont supplémentaires); de la même manière,

$$\frac{A'HC'}{ABC} = (1 + \cot A)(1 + \cot C), \quad \frac{B'HC'}{ABC} = (1 + \cot B)(1 + \cot C),$$

et additionnant ces équations,

$$\frac{A'B'C'}{ABC} = 3 + 2(\cot A + \cot B + \cot C) + (\cot A\cot B + \cot A\cot C + \cot B\cot C);$$

mais $$\cot A + \cot B + \cot C = \cot\alpha,$$

et $\cot A\cot B + \cot A\cot C + \cot B\cot C = 1$, $A + B + C$ étant $= 180°$,

donc $$\frac{A'B'C'}{ABC} = 3 + 2\cot\alpha + 1 = 4 + 2\cot\alpha,$$

d'où $\quad A'B'C' = 2ABC(2 + \cot\alpha)$, qui est l'équation (1).

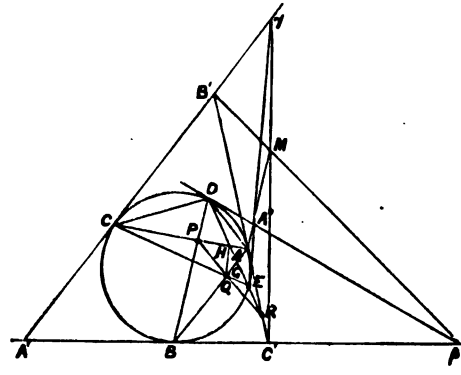
Des dernières formules obtenues on déduit

$$\cot\alpha' = \frac{A'B'^2 + B'C'^2 + A'C'^2}{4A'B'C'} = \frac{4ABC\,(3 + 2\cot\alpha)}{8ABC\,(2 + \cot\alpha)} = \frac{3 + 2\cot\alpha}{2\,(2 + \cot\alpha)}.$$

9057. (Emile Vigarié.)—Par les sommets d'un triangle donné ABC, on mène des tangentes a son cercle circonscrit qui déterminent le triangle A'B'C'. On joint les sommets A', B', C' à un point donné M. Les droites MA', MB', MC' coupent les tangentes menées par A, B, C, en α, β, γ. Enfin par les points α, β, γ on mène les tangentes $\alpha C''B''$, $\beta A''C''$, $\gamma B''A''$ au cercle circonscrit à ABC qui forment le triangle A''B''C''. Démontrer que les droites AA'', BB'', CC'' concourent au point donné M.

Solution by J. M. Stoops, M.A.; Sarah Marks, B.Sc.; *and others.*

Let the tangents from β, γ meet the circle in D, E. Join DE, and let it meet B'C' in R. Join AB, AC, CD, BD, CE. Let DE meet AC in H.



Join QH. Pencil Q . AHPC = range AHPC = pencil D . AHPC = an harmonic ratio of the four points on a circle; A, E, B, C = pencil A . AEBC = pencil A . REGH = range REGH = pencil Q . REGH = Q . GHRE; therefore PQR is a straight line.

Now, CE is the polar of γ, and BA the polar of C'. Hence Q is the pole of $\gamma C'$. So P is the pole of $\beta B'$. Thus M is the pole of PQ. DE is the polar of A''. AR is the polar of A. But the three polars PQ, DE, AR meet in a point, hence the three poles M, A'', A are in a right line. Therefore AA'' passes through M. So BB'' and CC'' pass through M.

8921. (H. L. Orchard, B.Sc., M.A.)—Solve the equation

$$2x^6 + 6x^5 + 5x^4 - x^3 - 53100 = 0.$$

Solution by R. F. Davis, M.A.; Prof. Beyens; *and others.*

If we put $U = x^2 + x$, the given equation may be written $2U^3 - U^2 = 53100$; one root of which is $U = 30$, and the other roots are given by the quadratic $2U^2 + 59U + 1770 = 0$. Hence $x = 5$ or $x = -6$ are the only real solutions.

8817. (W. J. Greenstreet, B.A.)—In the "ambiguous case," given a, b, and A; ν, ν' being radii of in-circles and ρ, ρ' radii of ex-circles to a, prove that $\nu\nu'(b+a) = \rho\rho'(b-a)$.

Solution by Asparagus; Prof. Beyens, M.A.; *and others.*

If c_1, c_2 be the values of the third side, c_1, c_2 are the roots of the quadratic in c, $c^2 - 2bc\cos A + b^2 - a^2 = 0$. Let K_1, K_2, denote double the areas of the two triangles, then we have

$$\nu = \frac{K_1}{a+b+c_1}, \quad \nu' = \frac{K_2}{a+b+c_2}, \quad \rho = \frac{K_1}{b+c_1-a}, \quad \rho' = \frac{K_2}{b+c_2-a};$$

hence $\nu\nu' : \rho\rho' = (b+c_1-a)(b+c_2-a) : (a+b+c_1)(a+b+c_2)$

$= (a-b)^2 - 2b(a-b)\cos A + b^2 - a^2 : (a+b)^2 + 2(a+b)b\cos A + b^2 - a^2$

$= (b-a)(b-a+b+a+2b\cos A) : (b+a)(b+a+b-a+2b\cos A)$

$= b-a : b+a.$

8736. (H. G. Dawson, M.A.)—P, P′, Q, Q′, R, R′ are six points on a conic; PP′, QQ′ meet in C; QQ′, RR′ in A; RR′, PP′ in B. Show that, if a quartic curve be described through P, P′, Q, Q′, R, R′, A, B, C, and if the tangents to it at ABC make angles α, α', β, β', γ, γ' with the sides AC, AB, BA, BC, CB, CA, then $\sin\alpha\sin\beta\sin\gamma = \sin\alpha'\sin\beta'\sin\gamma'$, and hence that a conic can be drawn through A, B, C touching the quartic in A, B, C.

Solution by the Rev. T. R. Terry, M.A.; Kate Gale; *and others.*

Let equation to conic be $a\alpha^2 + b\beta^2 + c\gamma^2 + 2a'\beta\gamma + 2b'\gamma\alpha + 2c'\alpha\beta = 0$ referred to ABC as triangle of reference, then equation to quadric is

$$P\beta\gamma(b\beta^2 + c\gamma^2 + 2a'\beta\gamma) + Q\gamma\alpha(\ldots) + R\alpha\beta(\ldots) = 0,$$

where P, Q, R are arbitrary constants.

The tangent to quadric at A is $Q\gamma + R\beta = 0$; therefore $\frac{Q}{\sin\alpha} = \frac{R}{\sin\alpha'}$,

whence $\sin\alpha\sin\beta\sin\gamma = \sin\alpha'\sin\beta'\sin\gamma'$. But it is well known that this last relation is satisfied by the tangents at A, B, C to any conic through these three points. Describe a conic through A, B, C touching the quadric at B and C. Then it follows that the tangent to the quadric at A is also tangent to the conic at A, *i.e.*, the conic and quadric touch at A, B, and C.

8936. (A. Russell, B.A.)—Prove that the respective equations to the ellipsoid and the wave surface may be written in the forms

$$\begin{vmatrix} 1-\frac{a^2}{x^2}, & 1, & 1 \\ 1, & 1-\frac{b^2}{y^2}, & 1 \\ 1, & 1, & 1-\frac{c^2}{z^2} \end{vmatrix} = 0, \quad \begin{vmatrix} \frac{a^2-y^2-z^2}{x^2}, & 1, & 1 \\ 1, & \frac{b^2-z^2-x^2}{y^2}, & 1 \\ 1, & 1, & \frac{c^2-x^2-y^2}{z^2} \end{vmatrix} = 0.$$

Solution by the Proposer.

Since $\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}-1=0$ may be written in the form

$$\begin{vmatrix} \theta+a^2-x^2, & -xy, & -xz \\ -yx, & \theta+b^2-y^2, & -yz \\ -zx, & -zy, & \theta+c^2-z^2 \end{vmatrix} = 0,$$

by putting $\theta=0$ we get the equation to the ellipsoid in this form; also, putting $a^2 \equiv -a^2$, and $\theta = r^2$, we get the equation to the wave surface in determinantal form.

8691. (B. Hanumanta Rau, B.A.)—Prove that

$$\int_0^{\frac{1}{2}\pi} \cos^6\theta \cos^2 5\theta \, d\theta = \frac{5\pi}{64}.$$

Solution by W. J. Greenstreet, B.A.; Rev. T. R. Terry, M.A.; *and others.*

$$\int_0^{\frac{1}{2}\pi} \cos^6\theta \cos^2 5\theta \, d\theta = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \cos^6\theta (1+\cos 10\theta)\, d\theta = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \cos^6\theta \, d\theta;$$

(since $\int_0^{\frac{1}{2}\pi} \cos^6\theta \cos 10\theta \, d\theta = 0$) $= \frac{1}{2}\cdot\frac{1.3.5}{2.4.6}\cdot\frac{\pi}{2} = \frac{5\pi}{64}$.

9059. (A. R. Johnson, M.A.)—In the cubic whose equation is

$$(x^2+y^2)(lx+my+n) = (l'x+m'y+n')^3,$$

show (1) that the product of the distances from the origin of the feet of the nine normals drawn from any point to the cubic, varies as the cube of the distance of the point from the origin; and find (2) what is the analogous theorem in the case of the quartic $(x^2+y^2)w = uv$, where $u=0$, $v=0$, $w=0$ represent any three conics, the two former having one common focus at the origin.

Solution by the Proposer.

Let $N_1, N_2, \ldots N_{n+m}$ be the feet of the normals drawn from a point P to a curve of order m and class n, and let O be any fixed point. Then the

product $ON_1^2 . ON_2^2 \dots ON_{n+m}^2$ is a rational integral function of the coordinates of P, whose degree is seen to be $2m$ by taking P at an infinite distance, when n of the distances ON remain finite, and m become infinite and of the order OP.

The product vanishes when P lies on any of the $2m$ normals at the points where OI and OJ meet the curve, and is therefore identical with the product of the left-hand members of the equations of the normals at the points. This product may be arranged as one of m real quadratic factors.

The same is true if, instead of the points N, we take the points Θ, where $P\Theta$ cuts the curve at a constant angle θ, on replacing the normals at the points of intersection of OI and OJ with the curve by lines through the points cutting the curve at an angle θ.

Now, suppose the curve and O such that the lines OI, OJ touch the curve at each of the points where they meet it, so that O is a multiple focus of the curve. Then, the line cutting the curve at any angle at the points of contact coincides with the tangent at the same point, *i.e.*, with OI or OJ. Taking O as origin, we have, then,

$$ON_1^2 . ON_2^2 \dots ON_{n+m}^2 = k\,(x+iy)^m\,(x-iy)^m,$$

or
$$ON_1 . ON_2 \dots ON_{n+m} = k\mathrm{OP}^m \dots\dots (1),$$

and in general
$$O\Theta_1 . O\Theta_2 \dots O\Theta_{n+m} = k'\mathrm{OP}^m \dots\dots (2).$$

The curves proposed in the question are particular cases. In the first, the origin is the intersection of two inflexional tangents which pass through the points I and J; in the second, it is the intersection of two bitangents which pass through I and J.

Now take P at infinity; then, n of the feet of the normals ($N_1, N_2 \dots N_n$) remain at a finite distance from O, while the other normals become the perpendiculars drawn from P on the asymptotes. Neglecting all but the highest powers of x and y, if $f(x, y) \equiv (y-\mu_1 x)\,(y-\mu_2 x) \dots (y-\mu_m x)$ be the terms of highest dimensions in the equation to the curve, we have

$$ON_{n+1} . ON_{n+2} \dots ON_{n+m} = \frac{(x+\mu_1 y)\,(x+\mu_2 y) \dots (x+\mu_m y)}{\left\{(1+\mu_1^2)\,(1+\mu_2^2) \dots (1+\mu_m^2)\right\}^{\frac{1}{2}}}$$
$$= \frac{f(y, -x)}{\left\{f(1, i) . f(1, -i)\right\}^{\frac{1}{2}}},$$

so that $ON_1 . ON_2 \dots ON_n = k\left\{f(1, i) f(1, -i)\right\}^{\frac{1}{2}}$ Lt. $\dfrac{(x^2+y^2)^{\frac{1}{2}m}}{f(y, -x)}$;

or, since $N_1, N_2, \dots N_n$ are now $T_1, T_2, \dots T_n$ say, the points of contact of tangents drawn in a direction making an angle ϕ with Ox, there results

$$OT_1 . OT_2 \dots OT_n = \frac{k\left\{f(1, i) f(1, -i)\right\}^{\frac{1}{2}}}{f(\cos\phi, \sin\phi)} \dots\dots (3).$$

This is a generalisation of the formula $SP . S'P = CD^2$ in conics. From (3) results at once

$$\sum_{r=1}^{r=n} \frac{C_r}{OT_r} = -\frac{d}{d\phi} \log f(\cos\phi, \sin\phi),$$

C_r denoting the chord of curvature at T_r perpendicular to OT_r.

On dividing (1) by (2), and making P move off to infinity, there results $k' = \frac{k}{\sin^m \theta}$, so that (1) is $O\Theta_1 . O\Theta_2 \ldots O\Theta_{n+m} = \frac{k}{\sin^m \theta} OP^m$.

Multiply by $\sin^m \theta$, and make $\theta = 0$. Then the n points $\Theta_1 \ldots \Theta_n$ become $T_1 \ldots T_n$, the points of contact of tangents drawn from P, and

$$O\Theta_{n+1} \sin\theta . O\Theta_{n+2} \sin\theta \ldots O\Theta_{n+m} \sin\theta$$

becomes A_p, the product of the perpendiculars drawn from P to the asymptotes. Hence $OT_1 . OT_2 \ldots OT_n . A_p = k . OP$ ..... (4), a generalisation of (3). This result for conics is well known, and comes at once from (4) by taking O at either focus.

8654. (F. Morley, M.A.)—There are n towns joined two and two by railroads. Show that it is possible to pass by rail over the entire system without using any twice if n be odd; and impossible if n be even.

Solution by the Proposer.

The thing is possible if, whenever we get to a town, we can get away by a new route, *i.e.*, if there is an even number of railways ending at that town, *i.e.*, if the whole number of towns is odd. Otherwise it is impossible.

9142. (R. W. D. Christie, M.A. See Quest. 8700.)—If

$$\Sigma_r = 1^r + 2^r + 3^r \ldots\ldots n^r,$$

prove that $(9\Sigma_{11} + 30\Sigma_9 + 9\Sigma_7)/\Sigma_3 = (11\Sigma_{10} + 30\Sigma_8 + 7\Sigma_6)/\Sigma_2$.

Solution by H. L. Orchard, B.Sc., M.A.; Sarah Marks; *and others.*

We find, by summing the series, that

$$9\Sigma_{11} + 30\Sigma_9 + 9\Sigma_7 = (3n^{12} + 18n^{11} + 45n^{10} + 60n^9 + 45n^8 + 18n^7 + 3n^6) / 4,$$

and that

$$11\Sigma_{10} + 30\Sigma_8 + 7\Sigma_6 = (6n^{11} + 33n^{10} + 75n^9 + 90n^8 + 60n^7 + 21n^6 + 3n^5) / 6.$$

Hence $$\frac{9\Sigma_{11} + 30\Sigma_9 + 9\Sigma_7}{11\Sigma_{10} + 30\Sigma_8 + 7\Sigma_6} = \&c. = \frac{3(n^2+n)}{2(2n+1)} = \frac{\Sigma_3}{\Sigma_2}.$$

[*Otherwise*:—We know (Quest. 8700) that $(n+1)\Sigma_n = \frac{d}{dx}\Sigma_{n+1}$,

$$\therefore \frac{9\Sigma_{11} + 30\Sigma_9 + 9\Sigma_7}{\Sigma_3} = \frac{9\frac{d}{dx}(\Sigma_{11}) + 30\frac{d}{dx}(\Sigma_9) + 9\frac{d}{dx}(\Sigma_7)}{3\frac{d}{dx}(\Sigma_3)} = \&c. = 48\Sigma_1^4\Big].$$

9197. (Professor HAUGHTON, F.R.S.)—If 2E denote the total sun-heat received by the whole Earth in a year, and if Δ denote the obliquity of the ecliptic, prove that each hemisphere receives, in a year,

$$\text{in summer, } \mathrm{E} \times \frac{\pi + 2\sin\Delta}{2\pi}; \quad \text{and in winter, } \mathrm{E} \times \frac{\pi - 2\sin\Delta}{2\pi}.$$

Solution by Professor F. X. DE WACHTER.

The common origin of summer and winter being at one of the equinoctial points, let ω denote its angular distance from the sun, the corresponding equatorial latitude of which $= \theta$ at the same time.

The heat received by one of the two hemispheres belongs to its winter-period, and that of the other hemisphere belongs to its summer-period. The quantity of heat received by the whole earth at the instant under consideration being $= \mathrm{H}\,d\omega$, then it follows that, for each hemisphere, $\frac{1}{2}\mathrm{H}(1+\sin\theta)\,d\omega$ and $\frac{1}{2}\mathrm{H}(1-\sin\theta)\,d\omega$ = elementary heat in the summer and winter periods; but, Δ being the obliquity of the ecliptic, we have $\sin\theta = \sin\Delta\sin\omega$; hence, by integration between $\omega = 0$ and $\omega = \pi$, we get at once, for each of the two hemispheres, $\frac{1}{2}\pi\mathrm{H} + \mathrm{H}\sin\Delta$, $\frac{1}{2}\pi\mathrm{H} - \mathrm{H}\sin\Delta$ = total heat received in summer and winter; now $\pi\mathrm{H} = \mathrm{E}$, and the above formulæ become $\mathrm{E}\,\frac{\pi + 2\sin\Delta}{2\pi}$ and $\mathrm{E}\,\frac{\pi - 2\sin\Delta}{2\pi}$.

2903. (Professor SYLVESTER, F.R.S.)—If i be any positive integer, and $(\Delta - 3)^i u_x = \dfrac{\Pi(2x)}{(\Pi x)^2}$, show that

$$u_x = (1, x)^{i-1}\,4^x + \frac{\Pi i \,.\, \Pi(2x+2i)}{\Pi 2i \,.\, \Pi x \,.\, \Pi(x+i)}.$$

[This theorem virtually contains a solution of the partial difference-equation (not noticed by Mr. BOOLE), $u_{x,y} = u_{x-1,y} + u_{x,y-1}$.]

Solution by Professor SEBASTIAN SIRCOM, M.A.

Let $u_x = 4^x v_x$, then $\Delta^i v_x = \dfrac{\Pi(2x)}{4^x(\Pi x)^2}$, whence $v_x = \Delta^{-i}\,\dfrac{\Pi(2x)}{4^x(\Pi x)^2}$. To integrate this, let $u_{x,y-1} = \dfrac{\Pi(y-1)\,.\,\Pi(2x+2y-2)}{4^x\Pi(2y-2)\,.\,\Pi x\,.\,\Pi(x+y-1)}$.

Then $$u_{x,y-1} = \frac{\Pi y \,.\, \Pi(2x+2y)}{4^x \,.\, \Pi 2y \,.\, \Pi x \,.\, \Pi(x+y)}\left(1 - \frac{2x}{2x+2y-1}\right),$$

whence, after reduction,

$$u_{x,y-1} = \frac{\Pi y \,.\, \Pi(2x+2y)}{4^x \,.\, \Pi 2y \,.\, \Pi x \,.\, \Pi(x+y)} - \frac{\Pi y \,.\, \Pi(2x-2+2y)}{4^{x-1} \,.\, \Pi 2y \,.\, \Pi(x-1) \,.\, \Pi(x-1+y)}$$

$$= u_{x,y} - u_{x-1,y},$$

so that $\Delta^{-1}u_{x-1,y} = \mathrm{C} + u_{x,y}$, by means of which we obtain the required result after restoring u_x.

9241. (Professor WOLSTENHOLME, M.A., Sc.D.)—1. The focus of a given parabola is S, and P is any point on the parabola; with focus S and vertex P is described another parabola meeting the former again in one real point Q; prove that PQ will be normal at P to the first parabola; and find the locus of the polar of PQ with respect to the second parabola.

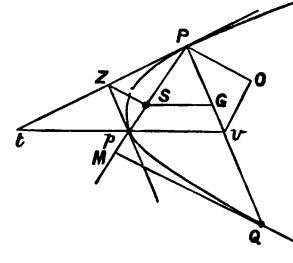
2. Prove by elementary geometry the following property of the circle, which is the reciprocal of the above theorem with regard to S. O is a fixed point on a given circle, OY perpendicular on the tangent to this circle at a point P; another circle is described on OY as diameter, and the second common tangent to the two circles meets the former tangent in R; prove that the angle POR is a right angle.

Solution by R. F. DAVIS, M.A.

1. This very interesting property is not easily proved by Cartesian coordinates, as there is considerable difficulty in obtaining the equation to the second parabola.

Polar coordinates are, however, better adapted for the purpose. If we take the equation to the first parabola as $2a/r = 1 + \cos\theta$ .....(i.), and $\angle ASP = \alpha$; the equations to the second parabola and to the normal chord PQ are then $4a/r = (1+\cos\alpha)\{1+\cos(\theta-\alpha)\}$ .....(ii.), and $2a\sin\frac{1}{2}\alpha/r = (1+\cos\alpha)\sin(\theta-\frac{1}{2}\alpha)$ .....(iii.).

Now it will be found that (ii.) and (iii.) both intersect (i.) again in the same point for which $\tan\frac{1}{2}\theta = -(3+\cos\alpha)\operatorname{cosec}\alpha$. The following geometrical proof is more simple and satisfactory. Let PSp be a focal chord, the tangents at whose extremities intersect at right angles on the directrix in Z, so that SZ is perpendicular to PSp. Then PQ, the chord of the parabola normal at P, is bisected in v by the diameter through p; and the tangent at P meets this diameter at a point t, such that $pv = pt$. Draw QM perpendicular to PSp.



Then QM : PQ = SZ : Zp.

But PQ = 2Pv = 4Zp; and QM = 4SZ.

Also QM : PM = SP : SZ = 4SP : QM;

therefore $QM^2 = 4SP \cdot PM$,

and Q lies on the parabola whose axis is PSp, vertex P, and focus S.

If O be the pole of PQ, it is the intersection of a line through v parallel to SP with a line through P perpendicular to SP.

Taking the coordinates of P as $(a/m^2, 2a/m)$, the coordinates of v are found to be $\{a(2m^2+2+m^{-2}), -2am\}$, and the equations to the lines described above are

$2am^4 - m^3y - 2m^2(x-3a) + my + 2a = 0$, $m^4x - 2m^3y - m^2(x-3a) + a = 0$,

from which m has with some difficulty to be eliminated.

These equations reduce to

$$m^2 \cdot x(x-a) - 2mxy - Y^2 = 0, \quad m^2 \cdot xy + 2mY^2 - xy = 0,$$

where $$Y^2 = a(x-a) - y^2.$$

Eliminating m, we find, after reduction, that the focus $\{(x-a)^2+y^2\}^2$ appears as an irrelevant factor, and that the final result is the cubic

$$y^2(x-4a) = 4a^2.$$

2. In this case R is the external centre of similitude of the two circles, and consequently $RO^2 = RY \cdot RP$; therefore, since OYP is a right angle, so also is POR.

[*Otherwise*:—(1) The normal at P to the parabola meeting the curve again in Q, draw PK, QL perpendicular to the directrix, and let S′ be the focus of a parabola having the same directrix, and vertex P. Then S′ lies on KP produced, and $PS' = KP = SP$; hence, if we join S′Q, the triangles S′PQ, SPQ will be equal in all respects, and $S'Q = SQ = QL$, or the parabola with focus S and vertex P passes through Q. Now suppose this parabola turned about the common chord PQ through an angle of 180°, then S′ will coincide with S, and the parabola will now be one having S for focus, P for vertex, and passing through Q.

Or again, using rectangular coordinates with S as origin, $ASP = 2\alpha = 2\tan^{-1}m$, the equations of the two parabolas will be

$$x^2+y^2 = (x+2a)^2, \quad x^2+y^2 = (x\cos 2\alpha + y\sin 2\alpha - 2a\sec^2\alpha)^2;$$

hence the equations of two common chords will be

$$x(1-\cos 2\alpha) - y\sin 2\alpha + 2a\tan^2\alpha = 0;$$

$$x(1+\cos 2\alpha) + y\sin 2\alpha = 2a(1+\sec^2\alpha);$$

and the former of these is the same as $y = m(x+a+am^2)$, which is normal at the point $x = a(1-m^2)$, $y = 2am$ to the parabola $y^2 = 4a(a-x)$.

The other common chord does not meet the parabolas in real points; its equation may be written $x+my = a(1+m^2)(2+m^2)$; and its envelope is of the fourth class and also of the fourth degree, having two cusps and one node, all of which are impossible.]

9245. (Professor Beyens.)—Démontrer par la méthode des equipollences la proposition bien connue: Si A_1, A_2, A_3 sont trois points pris sur le côté OX d'un angle XOY, et B_1, B_2, B_3 sont pris sur le côté OY, les points de rencontre M de A_2B_3, A_3B_2; N de A_3B_1, A_1B_3; P de A_1B_2, A_2B_1 sont en ligne droite.

Solution by Rev. D. Thomas, M.A.; G. G. Storr, M.A.; *and others.*

Let α, β be unit vectors along OA, OB respectively, and let α_1, α_2, α_3 be vectors measured from O of A_1, A_2, A_3, so that $l_1\alpha_1 = l_2\alpha_2 = l_3\alpha_3 = \alpha$, and β_1, β_2, β_3 for B_1, B_2, B_3, so that $m_1\beta_1 = m_2\beta_2 = m_3\beta_3 = \beta$, and if ρ_1, ρ_2, ρ_3 be the vectors of M, N, P, $\rho_1 = (1-x)\alpha_2 + x\beta_3 = (1-y)\alpha_3 + y\beta_2$, whence we easily get $\rho_1 = \{(m_3-m_2)\alpha + (l_3-l_2)\beta\}/(l_3m_3 - l_2m_2)$,

and similar values for ρ_2 and ρ_3. Then, because

$$(l_3m_3-l_2m_2)\rho_1 + (l_1m_1-l_3m_3)\rho_2 + (l_2m_2-l_1m_1)\rho_3 = 0,$$

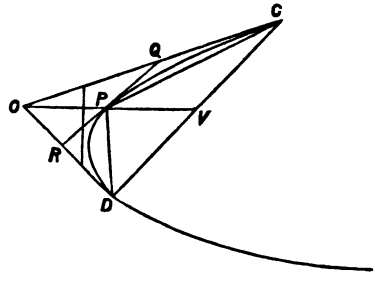
$$l_3m_3 - l_2m_2 + l_1m_1 - l_3m_3 + l_2m_2 - l_1m_1 = 0,$$

M, N, P are all on the same straight line.

8153. (R. KNOWLES, B.A., L.C.P.)—A diameter through the pole of CD, a chord of a parabola, meets the curve in P; prove that the area of the triangle PCD is twice that formed by joining the poles of its sides.

Solution by Rev. T. GALLIERS, M.A.; SARAH MARKS, B.Sc.; *and others.*

Let O be the pole of CD, OPV be parallel to the axis of the parabola and let the tangent to the curve at P meet OC, OD in Q, R. Then Q, R are the poles of PC, PD, and we know that QR is parallel to CD and OP = PV; therefore CD = 2QR; and perpendicular from O on QR = perpendicular from P on CD; therefore the area of the triangle CPD = twice the area of the triangle QOR.



9231. (Professor CROFTON, F.R.S.)—Prove that, if ξ be any operator containing only x, D, and χ any function of x,

$$\xi\chi = \chi\xi + \chi' \frac{d\xi}{d\mathrm{D}} + \frac{\chi''}{1.2}\frac{d^2\xi}{d\mathrm{D}^2} + \&c.,$$

χ', χ''... being derived.

[The theorem may be otherwise expressed as follows:—

$$\xi f(x) = f\left(x + \frac{d}{d\mathrm{D}}\right)\xi].$$

Solution by J. W. SHARPE, M.A.; Professor SIRCOM, M.A.; *and others.*

If we denote the operation $\frac{d}{d\mathrm{D}}$ by δ, the theorem to be proved may be written $\xi f(x) = f(x+\delta)\,\xi$, where δ operates only upon D, and x is a constant with respect to this operator. Let us take as representative a term in ξ, $\mathrm{D}^m x^r \mathrm{D}^n x^s$, it is to be proved that

$$\mathrm{D}^m x^r \mathrm{D}^n x^s f(x) = f(x+\delta)\,\mathrm{D}^m x^r \mathrm{D}^n x^s.$$

Consider first the operation $\mathrm{D}^n x^s$.

Now, if D_s, D_f denote differentiation with respect to x, performed upon x^s and $f(x)$ respectively, then we may write $\mathrm{D}^n x^s f(x) = (\mathrm{D}_s + \mathrm{D}_f)^n x^s f(x)$,

therefore $\mathrm{D}^n x^s f(x) = \left\{\mathrm{D}_s^n + n\mathrm{D}_s^{n-1}\mathrm{D}_f + \frac{n(n-1)}{1.2}\mathrm{D}_s^{n-2}\mathrm{D}_f^2 + \ldots\right\} x^s f(x)$,

and the general term is $\frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}\mathrm{D}_s^{n-r}\mathrm{D}_f^r x^s f(x)$.

Again, $f(x+\delta)\,\mathrm{D}^n x^s = \left\{f(x) + \delta f'(x) + \frac{\delta^2}{1.2}f''(x) + \&c.\right\}\mathrm{D}^n x^s$;

and the general term is

$$\frac{1}{1\,.\,2\,\ldots\,r}\,\frac{d^r f}{dx^r}\,\delta^r\,.\,\mathrm{D}^n x^s = \frac{n\,(n-1)\,\ldots\,(n-r+1)}{1\,.\,2\,\ldots\,r}\,\mathrm{D}_x^{n-r}\mathrm{D}_f^r\,x^s f(x);$$

whence it appears that the two expansions are identical, and therefore it follows that $\mathrm{D}^m x^r\,\mathrm{D}^n x^s f(x) = \mathrm{D}^m x^r f(x+\delta)\,\mathrm{D}^n x^s$.

Now let us denote the operation of differentiation performed upon D^m by δ_m, and that upon D^n by δ_n, then we have

$$f(x+\delta)\,\mathrm{D}^n x^s = f(x+\delta_n)\,\mathrm{D}^n\,(x+\delta_n)^s,$$

because $\mathrm{D}^n\,(x+\delta_n)^s = \mathrm{D}^n x^s$,

then
$$\begin{aligned}\mathrm{D}^m x^r f(x+\delta)\,\mathrm{D}^n x^s &= \mathrm{D}^m x^r f(x+\delta_n)\,\mathrm{D}^n\,(x+\delta_n)^s\\ &= \mathrm{D}^m x^r\,\{\text{function of } (x+\delta_n)\};\end{aligned}$$

but δ_n does not operate upon D^m; therefore we may apply the theorem already proved, and write

$$\begin{aligned}\mathrm{D}^m x^r f(x+\delta_n)\,\mathrm{D}^n x^s &= f(x+\delta_n+\delta_m)\,\mathrm{D}^n\,(x+\delta_n+\delta_m)^s\,\mathrm{D}^m x^r\\ &= f(x+\delta)\,\mathrm{D}^m x^r\,\mathrm{D}^n x^s,\end{aligned}$$

for $(\delta_n+\delta_m)\,\mathrm{D}^n\mathrm{D}^m = \delta\,.\,\mathrm{D}^n\mathrm{D}^m$.

This method of proof is obviously perfectly general, and will apply whatever be the number of terms $\mathrm{D}^m x^r$ in the general term of ξ. Hence the theorem follows.

[*Otherwise*:—By Leibnitz's theorem

$$\begin{aligned}\mathrm{D}^r f(x)\,y &\equiv f(x)\,\mathrm{D}^r y + r f'(x)\,\mathrm{D}^{r-1} y + r\,\frac{r-1}{2}\,f''(x)\,\mathrm{D}^{r-2} y + \ldots\\ &\equiv f\left(x+\frac{d}{d\mathrm{D}}\right)\mathrm{D}^r y.\end{aligned}$$

Also
$$\begin{aligned}\mathrm{D}^r x^s f(x)\,y &\equiv f(x)\,\mathrm{D}^r x^s y + r f'(x)\,\mathrm{D}^{r-1} x^s y + r\,\frac{r-1}{2}\,f''(x)\,\mathrm{D}^{r-2} x^s y + \ldots\\ &\equiv f\left(x+\frac{d}{d\mathrm{D}}\right)\mathrm{D}^r x^s y.\end{aligned}$$

If $r = m+n$, we have

$$\mathrm{D}^{m+n} f(x)\,y \equiv f\left(x+\frac{d}{d\mathrm{D}}\right)\mathrm{D}^{m+n} y \equiv \mathrm{D}^m f\left(x+\frac{d}{d\mathrm{D}}\right)\mathrm{D}^n y,$$

or the operators D^m and $f\left(x+\frac{d}{d\mathrm{D}}\right)$ are commutative. Then

$$\mathrm{D}^r x^s \mathrm{D}^p x^q f(x)\,y \equiv \mathrm{D}^r x^s f\left(x+\frac{d}{d\mathrm{D}}\right)\mathrm{D}^p x^q\,y = f\left(x+\frac{d}{d\mathrm{D}}\right)\mathrm{D}^r x^s\,\mathrm{D}^p x^q\,y.$$

Every term in ξ can be expressed in a similar manner, and adding the results we obtain the required theorem.]

9107. (F. Morley, M.A.)—Conics touch four fixed lines: show that (1) the points where they cut at right angles lie on a circular cubic through the six intersections of the lines and the three feet of perpendiculars of the diagonal triangle; and (2) the real asymptote of the cubic is parallel to the line that goes through the mid-points of the three diagonals.

Solution by Professor SCHOUTE.

1. The tangents to the two conics of the tangential that pass through an arbitrary point P in this point are the double rays of the involution formed by the couples of tangents through P to the conics of the faisceau or pencil. When for P these double rays are perpendicular to one another, the conic of the pencil that touches one of the two isotropic lines through P, also touches the other, *i.e.*, P is focus of one of the conics of the pencil. Therefore the curve in question is the locus of the foci of the conics that touch four given lines.

The resulting V cubic has been studied as locus of foci by Professor H. Schröter, of Breslau (*Mathematische Annalen*, t. v., p. 50, "Ueber eine besondere Curve 3ter Ordnung und eine einfache Erzeugungsart der allgemeinen Curve 3ter Ordnung"), and as locus of points of brightness by Professor C. Pelz, of Graz (*Sitzungsberichte der Math.-naturw. Classe der Kais. Akad. d. Wissensch. in Wien*, t. lxiv., p. 730, 1871, "Ueber das Problem der Glanzpunkte"). The first memoirs contain the proof of (1).

2. Here the real asymptote does not itself pass through the mid-points of the three diagonals, but is only parallel to the line containing these points. When P is the focus of the parabola belonging to the pencil, and l the join of the three mid-points, the asymptote and P lie on opposite sides of l and at the same distance. This is immediately evident, when l is considered as the locus of the locus of the mid-point of any segment limited by a couple of conjugate points of the cubic, etc.

9211. (The EDITOR.)—A straight rod, divided into $2n$ equal parts, is broken at random into three parts at the points of division; find the probability that these three parts will form (1) a triangle of any kind, (2) an acute-angled triangle.

Solution of (1) *by* Professor F. X. DE WACHTER.

Construct the equilateral triangle whose altitude equals length of the rod, and divide each of its sides in $2n$ equal parts. From those points of division if parallels be drawn to the sides, they will intersect each other within the triangle in $(n-1)(2n-1)$ points. Among those points of intersection there will be $\frac{1}{2}(n-2)(n-1)$ favorable ones, and the required probability is $\frac{n-2}{2(2n-1)}$.

9199. (Professor HUDSON, M.A.)—Given an angle of a triangle, the direction of the sides containing it, and the circum-radius; find the locus of the orthocentre.

Solution by Professors PLAMENEWSKY, BEYENS, *and others.*

Pour la distance de l'orthocentre du sommet A du triangle nous avons $2R\cos A$, où $\angle A$ est donné. Donc le lieu de l'orthocentre est une circonférence dont le rayon est égal à $2R\cos A$ et le centre dans le sommet A.

9240. (Professor NILKANTHA SARKAR, M.A.)—If a triangle be inscribed in a circle of radius a, prove that its mean area is $3a^2/2\pi$.

Solution by R. F. DAVIS, M.A.

Let TOT′ be a tangent to the circle at O, and let OP, OQ be drawn making angles θ, ϕ with OT, OP respectively.

Then the limits of ϕ are 0 to $\pi-\theta$ (when θ is fixed),

,, ,, θ ,, 0 to π;

and using the formula $2R^2 \sin A \sin B \sin C$ for the area of a triangle, we have the mean value of the triangle POQ

$$= 2a^2 \int_0^{\pi} \int_0^{\pi-\theta} \sin\theta \sin\phi \sin(\theta+\phi)\, d\theta\, d\phi \Big/ \int_0^{\pi} \int_0^{\pi-\theta} d\theta\, d\phi = \frac{3a^2}{2\pi}.$$

9156. (Professor NEUBERG.)—Soient a, b, c, d les hauteurs d'un tétraèdre ABCD ; $\alpha, \beta, \gamma, \delta$ les angles des faces avec un plan P ; p la distance du centre de la sphère ABCD à P. Démontrer que tout point M de P satisfait à la relation

$$\frac{\cos\alpha}{a}(AM)^2 + \frac{\cos\beta}{b}(BM)^2 + \frac{\cos\gamma}{c}(CM)^2 + \frac{\cos\delta}{d}(DM)^2 = 2p \quad \ldots\ldots(1).$$

Solution by the PROPOSER.

Soient, par rapport à trois axes rectangulaires menés par le centre O de la sphère ABCD, (x_1, y_1, z_1), ... les coordonnées de A, B, C, D. L'équation

$$\Sigma \lambda_1 \{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2\} = 2p \quad \ldots\ldots(2)$$

peut représenter le plan P, pourvu que $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ vérifient les relations

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0, \quad \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 = -\cos f,$$

$$\lambda_1 y_1 \ldots\ldots = -\cos g, \quad \lambda_1 z_1 + \ldots\ldots = -\cos h.$$

f, g, h étant les angles de P avec les plans des coordonnées. De là on tire :

$$\lambda_1 = -\begin{vmatrix} 0 & 1 & 1 & 1 \\ \cos f & x_2 & x_3 & x_4 \\ \cos g & y_2 & y_3 & y_4 \\ \cos h & z_2 & z_3 & z_4 \end{vmatrix} : \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

L'équation du plan BCD étant

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x & x_2 & x_3 & x_4 \\ y & y_2 & y_3 & y_4 \\ z & z_2 & z_3 & z_4 \end{vmatrix} \equiv xX + yY + zZ + U = 0,$$

et les angles de BCD avec les plans des coordonnées étant f', g', h', on trouve

$$\cos f' = \frac{X}{(X^2+Y^2+Z^2)^{\frac{1}{2}}}, \quad \ldots, \quad a = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix} (X^2+Y^2+Z^2)^{-\frac{1}{2}};$$

par conséquent $\lambda_1 = -\dfrac{\cos f \cos f' + \cos g \cos g' + \cos h \cos h'}{a} = \dfrac{\cos a}{a}$.

Si l'on porte les valeurs dans l'équation (2), on a la relation proposée (1).

Corollaire.—Si M désigne un point non situé dans le plan P, la distance de M à P est égale à la demi-différence des deux membres de (1).

8744. (D. Edwardes.)—A sphere of radius a is surrounded by an infinite mass of incompressible fluid. If the surface of the sphere be suddenly moved with normal velocity $eyz + fzx + gxy$ at any point, prove that the velocity potential of the resulting initial motion is $-a^6(eyz + fzx + gxy)/(3r^5)$, where r is the distance of any point from the centre.

Solution by Rev. T. R. Terry, M.A.; Professor Beyens; *and others.*

If ϕ be the velocity potential, then clearly ϕ must be of the form $\phi = Q_2/r^3$, where $r^2Q_2 = Ax^2 + By^2 - (A+B)z^2 + Eyz + Fzx + Gxy$.

When $r = a$, $d\phi/dr = eyz + fzx + gxy$; therefore, equating coefficients,

$$A = B = 0, \quad -3E/a^6 = e, \text{ \&c.},$$

therefore $$\phi = -a^6(eyz + fzx + gxy)/(3r^5).$$

8703. (S. Tebay, B.A.)—The distances x, y, z of a point O from the angles of a triangle form the base of a tetrahedron whose conterminous edges a, b, c are respectively opposite to x, y, z. Find the position of O when the volume is a maximum.

Solution by the Proposer.

Let $$s_1 = b^2 + c^2 - a^2, \quad s_2 = c^2 + a^2 - b^2, \quad s_3 = a^2 + b^2 - c^2,$$
$$x' = a^2 - x^2, \quad y' = b^2 - y^2, \quad z' = c^2 - z^2;$$

then we have (see Quest. 8639)

$$x'y'z' = 4a^2b^2c^2 - a^2(s_1 + x')^2 - b^2(s_2 + y')^2 - c^2(s_3 + z')^2 + (s_1 + x')(s_2 + y')(s_3 + z') \quad \ldots\ldots\ldots\ldots (A),$$

therefore $$y'z'\delta x' + z'x'\delta y' + x'y'\delta z' = 0,$$

$$\{(s_2 + y')(s_3 + z') - 2a^2(s_1 + x')\}\,\delta x' + \{(s_3 + z')(s_1 + x') - 2b^2(s_2 + y')\}\,\delta y' + \{(s_1 + x')(s_2 + y') - 2c^2(s_3 + z')\}\,\delta z' = 0.$$

Multiply the first by λ, and add; compare the coefficients of $\delta x'$, $\delta y'$, $\delta z'$, and then eliminate λ; we thus obtain

$$x'(s_2s_3 - 2a^2s_1 - 2a^2x' + s_3y' + s_2z') = y'(s_3s_1 - 2b^2s_2 - 2b^2y' + s_1z' + s_3x') = z'(s_1s_2 - 2c^2s_3 - 2c^2z' + s_2x' + s_1y').$$

These and equation (A) determine the point O.

8429. (H. J. Read, B.A.)—Transform, by means of the geometry of the circle, the integrals

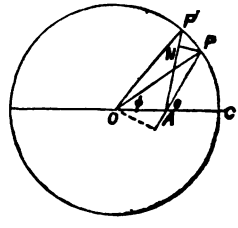
$$\int_0^{2\pi} \frac{d\phi}{(1+a^2-2a\cos\phi)^{\frac{1}{2}}}, \quad \int_0^{2\pi} \frac{\cos\phi\, d\phi}{(1+a^2-2a\cos\phi)^{\frac{1}{2}}}.$$

(See Cheyne's *Planetary Theory*, p. 50 *sqq.*)

Solution by the Proposer; Belle Easton, B.Sc.; *and others.*

Take a circle of radius unity. Let

$$OA = a, \quad \angle AOP = \phi, \quad \angle CAP = \theta.$$



Then $$\frac{OP}{OA} = \frac{\sin\theta}{\sin(\theta-\phi)},$$

or $$\sin(\theta-\phi) = a\sin\theta.$$

Also $$AP^2 = OP^2 + OA^2 - 2OP \cdot OA\cos\phi$$
$$= 1 + a^2 - 2a\cos\phi.$$

Take P' close to P, and draw PN perpendicular to AP'.

Then $$PN = PP'\sin NP'P \quad \text{or} \quad AP\,d\theta = \cos(\theta-\phi)\,d\phi.$$

Then $$\frac{d\phi}{AP} = \frac{d\theta}{\cos(\theta-\phi)} = \frac{d\theta}{(1-a^2\sin^2\theta)^{\frac{1}{2}}},$$

or $$\frac{d\phi}{(1+a^2-2a\cos\phi)^{\frac{1}{2}}} = \frac{d\theta}{(1-a^2\sin^2\theta)^{\frac{1}{2}}} \quad \ldots\ldots(1).$$

Again, $$AP = PL - AL,$$

or $$(1+a^2-2a\cos\phi)^{\frac{1}{2}} = \cos(\theta-\phi) - a\cos\theta = (1-a^2\sin^2\theta)^{\frac{1}{2}} - a\cos\theta.$$

Reducing, we get $$\cos\phi = a\sin^2\theta + \cos\theta\,(1-a^2\sin^2\theta)^{\frac{1}{2}} \quad \ldots\ldots(2).$$

From (1) the first integral transforms into $4\,\mathrm{F}(a\,.\,\frac{1}{2}\pi)$. From (2) the second integral transforms into $4\left\{\mathrm{F}(a\,.\,\frac{1}{2}\pi) - \mathrm{E}(a\,.\,\frac{1}{2}\pi)\right\}$.

9207. (Professor Bordage.)—If a and b are perfect numbers (*nombres entiers*), $a^2b\,(a^3+2a^2-a-2)(a^4-b^4)$ is divisible by 720.

Solution by R. W. D. Christie, M.A.; Prof. Plamenewsky; *and others.*

We know that (a^4-b^4) is always divisible by 10; and $(a+1)$ is never divisible by 3. Therefore

$$a^2b\,(a^3+2a^2-a-2)(a^4-b^4) = (a-1)(a)(a+1)(a+2)(ab)(a^4-b^4)$$

has for its factors either $(3)(2)(a+1)(3)(4)(10)$ or $(a-1)(6)(a+1)(2)(6)(10)$. In either case the given expression is divisible by 720 *at least;* and it is easy to show that a larger number must be a factor.

8730. (J. Brill, M.A.)—If the triangles ABC and DEF have the same centroid, prove that

$$\frac{AD.AE.AF.BC}{\sin(BDC+BEC+BFC-BAC)} = \frac{BD.BE.BF.CA}{\sin(CDA+CEA+CFA-CBA)} = \frac{CD.CE.CF.AB}{\sin(ADB+AEB+AFB-ACB)},$$

where BDC denotes the angle through which DB must be turned in the positive direction in order to coincide with DC, and so on.

Solution by the Proposer; Professor Sarkar, M.A.; *and others.*

We have $(x-a)(y-a)(z-a)(b-c)+(x-b)(y-b)(z-b)(c-a)$
$+(x-c)(y-c)(z-c)(a-b)+(x+y+z-a-b-c)(b-c)(c-a)(a-b)\equiv 0.$
If we write $x+y+z=a+b+c$, this identity becomes

$$(x-a)(y-a)(z-a)(b-c)+(x-b)(y-b)(z-b)(c-a)+(x-c)(y-c)(z-c)(a-b)=0.$$

Let O be any point in the plane of the two triangles; then, since they have a common centroid, we have

$$(OA)+(OB)+(OC)=(OD)+(OE)+(OF),$$

and therefore

$$(AD).(AE).(AF).(BC)+(BD).(BE).(BF).(CA)+(CD).(CE).(CF).(AB)=0.$$

If we interpret this according to the rules laid down in my paper "On the Application of the Theory of Complex Quantities to Plane Geometry," (*Messenger of Mathematics*, Vol. xvi., 1886), we shall obtain the above result.

9158. (Munindranath Bhattacharyya)—Show that the equation to the n^{th} pedal of the curve $r=f(p)$ is $\frac{r^{n+1}}{p^n}=f\left(\frac{r^n}{p^{n-1}}\right)$.

Solution by Beni Madhav Sarkar, B.A.; A. Kahn, M.A.; *and others.*

Substituting p^2/p' for r, and putting p and r for p' and p, respectively,

$$r^2/p=f(r) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

is the equation to the 1st pedal. Making a similar substitution in (1),

$$\frac{r^3}{p^2}=f\left(\frac{r^2}{p}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

is the equation to the 2nd pedal. In the same way,

$$\frac{r^4}{p^3}=f\left(\frac{r^3}{p^2}\right) \ \ldots \text{ is the equation to the 3rd pedal,}$$

and by an easy induction the equation to the n^{th} pedal is

$$\frac{r^{n+1}}{p^n}=f\left(\frac{r^n}{p^{n-1}}\right).$$

9159. (Professor WOLSTENHOLME, M.A., Sc.D.)—Any point P of a certain curve being taken, the corresponding points of an involute and of the evolute are Q, Q′; prove that, if QQ′ pass always through a fixed point O, the curve will have for its intrinsic equation $s = a \sin m\psi$ or $s = Ae^{m\psi} + Be^{-m\psi}$; also OQ : OQ′ will be a constant ratio.

Solution by R. F. DAVIS, M.A.

Let P be any point on a curve; Q the point on the evolute corresponding to P; R the point on the evolute of the evolute corresponding to Q.

Then, if ψ be the angle which the tangent at P makes with a fixed line,

$$PQ = \rho = \frac{ds}{d\psi}, \quad QR = \frac{d\rho}{d\psi}, \text{ and } \tan QPR = \frac{1}{\rho}.\frac{d\rho}{d}.$$

Wherefore, if PR always passes through O,

$$\tan QPR = \cot OPY, \text{ or } \frac{1}{\rho}.\frac{d\rho}{d\psi} = \frac{1}{p}.\frac{dp}{d\psi};$$

where $OY = p$.

Integrating, $\log \rho = \log p + \text{const.}$, or $\rho \propto p$.

But $\rho = p + \frac{d^2p}{d\psi^2}$, hence $\frac{d^2p}{d\psi^2} \propto p$, &c.

Also $OP : PR = p : \rho = \text{constant}$.

[In this solution the notation used is slightly different from that of Professor WOLSTENHOLME.]

9201. (Professor MATZ, M.A.)—P is a point within a triangle ABC, and AP, BP, CP meet the opposite sides in A′, B′, C′, respectively; if Pa, Pb, Pc be measured along PA, PB, PC so that these last are harmonic means between PA′, Pa; PB′, Pb; PC′, Pc, respectively, prove that a, b, c lie on a straight line.

Solution by Rev. D. THOMAS, M.A.; Prof. SARKAR, M.A.; *and others.*

Let $m : n$, $n : l$, $l : m$ be the ratios in which BC, CA, AB are divided at A′B′C′ respectively; and α, β, γ vectors of A, B, C measured from a point O; then the vector of P will be $(mn\alpha + nl\beta + lm\gamma) / (mn + nl + lm)$, and, by the Question,

$$Pa = PA \,.\, PA' / (2PA' - PA)$$
$$= PA\, mn / (2mn - nl - lm);$$

because $PA : PA' = l(m+n) : mn$;

then Pa as vector $= Pa\,(OA - OP) / PA$

$$= lmn \left\{(m+n)\alpha - n\beta - m\gamma\right\} / (2mn - nl - lm) \,.\, \Sigma mn,$$

and like values for Pb and Pc. Then a, b, c lie on a line, because

$$\Sigma (2mn - nl - lm)\, Pa = 0 = \Sigma (2mn - nl - lm).$$

8140. (H. J. Read, B.A.)—Show that the area of a quadrilateral is half of the value of the determinant

$$\begin{vmatrix} x_1 & y_1 & 10 \\ x_2 & y_2 & 11 \\ x_3 & y_3 & 10 \\ x_4 & y_4 & 11 \end{vmatrix}.$$

Solution by the Proposer; Sarah Marks, B.Sc.; *and others.*

For the determinant becomes, on expansion,

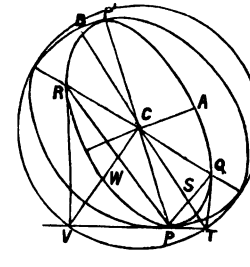
$$x_1(y_2-y_4)+x_2(y_3-y_1)+x_3(y_4-y_2)+x_4(y_1-y_3).$$

From this can be obtained the expression for the area of a quadrilateral in a circle, or in an ellipse, as in Salmon's *Higher Algebra*, in the case of the area of a triangle. It is probable that the area of any polygon may be similarly expressed.

9161. (Professor Hudson, M.A.)—If the semi-axes of an ellipse are CA, CB; prove that the major axes of all ellipses touching the given ellipse and having their foci at the ends of a diameter thereof are equal and equal to 2AB.

Solution by Professor F. X. De Wachter, M.A.

Let R and Q be the extremities of any diameter in the given ellipse and the foci of the enveloping one. Both ellipses being concentric, their points of contact lie on a common diameter PCP'.



Now, PR and PQ are supplemental chords, and equally inclined on the tangent at P. Therefore, if the diameters parallel to RP and PQ be produced to meet that tangent respectively in T and V, then VCT is an isosceles triangle and CT bisects PQ in S, CV bisecting PR in W. And because the four angles at the base TPV are equal to one another, it follows that

$$RW = WP = VW \quad \text{and} \quad SQ = SP = ST;$$

whence $RP + PQ = 2(WP + PS) = 2(SC + ST) = 2CT = 2CV.$

Besides $\widehat{PTQ}$ and $\widehat{RVP}$ are right angles, which results also from the above equalities. But, CT being a conjugate direction to PQ, and TP a tangent, T is the pole of PQ; hence TQ is another tangent at Q. Therefore the locus of T (and of V) is that of the vertex of a right angle circumscribed to the ellipse, viz., a circle described about the rectangle of the axes 2CA and 2CB, that rectangle being circumscribed to the given ellipse (*i.e.*, the director-circle.) The radius of the circle is therefore equal to AB. Hence the major axis of the touching ellipse, or

$$PR + PQ = 2CT = \text{diameter } 2AB.$$

8599. (Professor SIRCOM, M.A.)—If $\frac{d}{dx} \equiv \mathrm{D}$, prove that

$$\left\{x^{-pr}\,(x^{p+1}\mathrm{D})^r\,x\right\}^p \phi(x) = x^p\,\mathrm{D}^{pr}\left\{x^{pr}\,\phi(x)\right\}.$$

Solution by the PROPOSER; H. J. READ, B.A.; *and others.*

(1) The operator in question follows the commutative law, for it is a function of $x\,\frac{d}{dx}$. For $x^{p+1}\frac{d}{dx}(x^q y) = x^{p+q}\left(x\frac{d}{dx}+q\right)y$,

$$\left(x^{p+1}\frac{d}{dx}\right)^2 x^q y = x^{p+1}\frac{d}{dx}\left\{x^{p+q}\left(x\frac{d}{dx}+q\right)\right\}y$$

$$= x^{2p+q}\left(x\frac{d}{dx}+p+q\right)\left(x\frac{d}{dx}+q\right)y.$$

Proceeding thus, we find

$$\left(x^{p+1}\frac{d}{dx}\right)^r x^q y = x^{pr+q}\left\{x\frac{d}{dx}+(r-1)p+q\right\}\ldots\left(x\frac{d}{dx}+q\right)y;$$

hence $x^{-pr-q}\left(x^{p+1}\frac{d}{dx}\right)^r x^q$ follows the commutative law.

(2) Taking, for instance, $p = 3$, we have

$$x^{-3r}\left(x^4\frac{d}{dx}\right)^r x\,.\,x^{-3r}\left(x^4\frac{d}{dx}\right)^r x\,.\,x^{-3r}\left(x^4\frac{d}{dx}\right)^r x\,.\,\phi(x)$$

$$= x^3\,.\,x^{-3r-3}\left(x^4\frac{d}{dx}\right)^r x^3\,.\,x^{-3r-3}\left(x^4\frac{d}{dx}\right)^r x^2\,.\,x^{-3r-1}\left(x^4\frac{d}{dx}\right)^r x\,.\,\phi(x),$$

which, by the preceding, becomes

$$x^3\left\{x\frac{d}{dx}+3(r-1)+3\right\}\ldots\left(x\frac{d}{dx}+3\right)\left\{x\frac{d}{dx}+3(r-1)+2\right\}$$

$$\ldots\left(x\frac{d}{dx}+2\right)\left\{x\frac{d}{dx}+3(r-1)+1\right\}\ldots\left(x\frac{d}{dx}+1\right)\phi(x)$$

$$= x^3\left(x\frac{d}{dx}+3r\right)\left(x\frac{d}{dx}+3r-1\right)\ldots\left(x\frac{d}{dx}+2\right)\left(x\frac{d}{dx}+1\right)\phi(x)$$

$$= x^3\,.\,x^{-3r}\,x\frac{d}{dx}\left(x\frac{d}{dx}-1\right)\ldots\left(x\frac{d}{dx}-3r+1\right)x^{3r}\,\phi(x)$$

$$= x^3\,.\,x^{-3r}\,.\,x^{3r}\left(\frac{d}{dx}\right)^{3r} x^{3r}\,\phi(x) = x^3\left(\frac{d}{dx}\right)^{3r} x^{3r}\,\phi(x).$$

Similarly, we can prove the result for any value of p.

8610. (S. TEBAY, B.A.)—If $\Sigma_1^\infty \frac{1}{i^{2n}} = c_{2n}\,u^{2n}$, and a_{2n} be the coefficient of θ^{2n} in the expansion of $\theta\cot\theta$, show that $a_{2n} + 2c_{2n} = 0$.

Solution by the PROPOSER; Professor MATZ, M.A.; *and others.*

Let $\frac{1}{n!}$ be denoted by P_n, then we have

$$\sin\theta = \theta - P_3\theta^3 + P_5\theta^5 - \&c., \quad \cos\theta = 1 - P_2\theta^2 + P_4\theta^4 - \&c.$$

The roots of $\sin\theta = 0$ are $0, \pm\pi, \pm 2\pi, \pm 3\pi$, &c., therefore

$$0 = (\theta^2 - \pi^2)(\theta^2 - 4\pi^2)(\theta^2 - 9\pi^2).....$$

Now, if a, b, c, &c. be the roots of the equation

$$0 = A_0 + A_1 x + A_2 x^2 + ...$$

we have $\Sigma\frac{1}{a} = -\frac{A_1}{A_0}$, $\Sigma\frac{1}{ab} = \frac{A_2}{A_0}$, $\Sigma\frac{1}{abc} = -\frac{A_3}{A_0}$, &c.,

and, if $C_n = a^{-n} + b^{-n} + c^{-n} + \&c.$, it can be shown that

$$A_0 C_n = A_1 C_{n-1} + A_2 C_{n-2} + ... + n(-1)^{n-1} A_n;$$

therefore $C_{2n} = P_3 C_{2n-2} - P_5 C_{2n-4} + ... + n(-1)^{n-1} P_{2n+1}$.

Again, $\theta\cot\theta = \frac{1 - P_2\theta^2 + P_4\theta^4 - \&c.}{1 - P_3\theta^2 + P_5\theta^4 - \&c.} = 1 + a_2\theta^2 + a_4\theta^4 + ...$ suppose.

Multiplying out, and comparing coefficients, we have

$$(-1)^n P_{2n} = a_{2n} - P_3 a_{2n-2} + P_5 a_{2n-4} + ... + (-1)^n P_{2n+1};$$

and, observing that

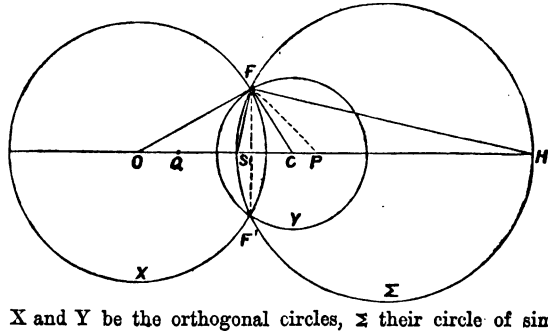
$$(-1)^n\{P_{2n} - P_{2n+1}\} = 2n(-1)^n P_{2n+1} = -2n(-1)^{n-1} P_{2n+1},$$

we have $a_{2n} = P_3 a_{2n-2} - P_5 a_{2n-4} + ... - 2n(-1)^{n-1} P_{2n+1} = -2C_{2n}$,

therefore $a_{2n} + 2C_{2n} = 0$.

8896. (F. MORLEY, B.A.)—Prove that, (1) the inverse of either centre of similitude of two orthogonal circles with respect to either circle is the inverse of the other centre with respect to the other circle; and (2) the two inverses and the two intersections of the circles are the angular points of a square.

Solution by D. O. S. DAVIES; Professor SARKAR, M.A.; *and others.*



Let X and Y be the orthogonal circles, Σ their circle of similitude.

Let P be the inverse of the external centre of similitude H with regard to the circle Y whose centre is C. Then $C\widehat{F}P = C\widehat{H}F = S\widehat{F}F'$. Since S, H are harmonic conjugates to O, C, and the angles OFC, SFH right, then OFS, SFC, CFH are each half a right angle. Hence $F'\widehat{F}P = \frac{1}{2}$ right angle; therefore $S\widehat{P}F = \frac{1}{2}$ right angle $= S\widehat{F}O$. Therefore P is the inverse of S the internal centre of similitude with regard to the circle X. If Q be the other inverse point, it follows that $F\widehat{Q}P = \frac{1}{2}$ right angle $= F\widehat{P}Q$. Hence F, F', P, Q are the angular points of a square.

9140. (Emile Vigarié.)—Si R, R_1, R_2 désignent respectivement les rayons du cercle circonscrit, du premier cercle de Lemoine (*triplicate ratio circle*) et le deuxième cercle de Lemoine (*cosine circle*); démontrer la relation $R^2 = 4R_1^2 - R_2^2$.

9202. (Professor Ignacio Beyens, M.A.)—Si R_1, R_2, ρ sont les rayons du premier cercle de Lemoine, du deuxième cercle de Lemoine, et du cercle de Brocard, on aura: $\rho^2 = R_1^2 - R_2^2$.

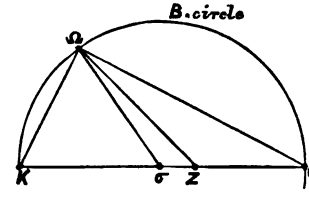
9184. (Emile Vigarié. Suggested by Question 9140.)—Si le segment OK qui joint le centre du cercle circonscrit au point de Lemoine (*Symmedian point*) est divisé dans le rapport $p : q$ par le centre d'un cercle de Tucker, et si R_1, R_2 sont respectivement les rayons du premier cercle de Lemoine et du deuxième cercle de Lemoine, le rayon P du cercle de Tucker considéré sera donné par la formule

$$P = [4q^2R_1^2 + (p^2 - q^2)R_2^2]^{\frac{1}{2}} / (p+q).$$

Solution by R. F. Davis, M.A.

Let R, ω be the circum-radius and B. angle of a triangle. Then the T. R. circle has its centre at σ the mid-point of OK, and its radius $R_1 = \frac{1}{2}R\sec\omega$. The C. circle has its centre at K and its radius $R_2 = R\tan\omega$. Hence

$$R^2 = R^2(\sec^2\omega - \tan^2\omega) = 4R_1^2 - R_2^2.$$



The B. circle has its centre at σ and radius $\rho = \frac{1}{2}OK$. Hence $R_1^2 - \rho^2$ = product of segments of chord of T. R. circle through $K = R^2\tan^2\omega = R_2^2$.

Let Z be the centre of a T. circle dividing OK in the ratio OZ : KZ $= p : q$, whose radius is denoted by P. Then, if Ω be a B. point, and consequently $O\Omega K$ a right angle, we have

$$(p+q)^2 Z\Omega^2 = q^2 . O\Omega^2 + p^2 . K\Omega^2.$$

But $$O\Omega : K\Omega : Z\Omega = R : R_2 : P,$$

therefore $$(p+q)^2 P^2 = q^2R^2 + p^2R_2^2 = p^2R_2^2 + q^2(4R_1^2 - R_2^2).$$

7189. (Professor SYLVESTER, F.R.S.)—Sum the series

$$1+(x-i)+\frac{(x-2i)(x-2i-1)}{1.2}+\frac{(x-3i)(x-3i-1)(x-3i-2)}{1.2.3}+\ldots.$$

Solution by Professor SEBASTIAN SIRCOM, M.A.

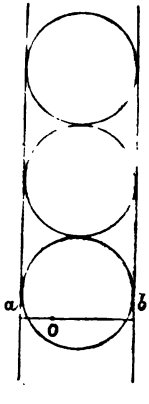
Expanding $\{1-z(1+z^i)\}^{-1}$ in powers of $z(1+z^i)$, we see that the term involving z^x arises from terms of the form $z^{x-ri}(1+z^i)^{x-ri}$; and putting $r=0, 1, 2$, &c., the given series is seen to be the coefficient of z^x in this expansion. Hence let α be a root of the equation $\phi(y)\equiv y^{i+1}y^i-1=0$, where $y=\frac{1}{z}$, then $\frac{1}{1-z-z^{i+1}}=\Sigma\frac{\alpha^i}{\phi'(\alpha(1-\alpha z)}$, and the coefficient of z^x will therefore be $\Sigma\frac{\alpha^{x+i}}{\phi'(\alpha)}$, which is the required sum.

If $i=1$, the sum is $\frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^{x+1}-\frac{1}{\sqrt{5}}\left(\frac{-\sqrt{5}+1}{2}\right)^{x+1}$, as may easily be verified.

[The PROPOSER's solution, with a slightly different result, is given on p. 32 of Vol. XL.].

8738. (Rev. T. R. TERRY, M.A.)—AOB is a straight line, AO = a, OB = b. With O as focus, a family of conics is described passing through the points A, B, each conic touching the preceding member of the family. If e is the eccentricity, show that the values of the expression $[(a+b)^2e^2-(a-b)^2]^{\frac{1}{2}}$ for the different conics form an arithmetic progression, whose common difference is $2(a+b)$.

Solution by the PROPOSER.



The figure represents a series of circles touching two parallel lines, and each touching the next member of the series.

Let $Oa=\frac{k^2}{a}$, $Ob=\frac{k^2}{b}$, and h = height of centre of first circle above line ab.

Reciprocate the whole figure with respect to a circle centre O and radius k. If e be the eccentricity of the n^{th} conic, $[(a+b)^2e^2-(a-b)^2]^{\frac{1}{2}}=2h+2n(a+b)$, whence the required result.

2906. (Professor SYLVESTER, F.R.S.)—Prove that (1) the continued fraction $1+\frac{1}{p}\frac{1}{+q}\frac{1}{+r}+\ldots$, where p, q, r are the terms of the natural harmonic progression 1, $\frac{1}{2}$, $\frac{1}{3}$, ... is equal to $\frac{1}{2}\pi$; and (2), in employing the successive convergents to the above continued fraction, explain the advantage of leaving off with a numerator 1 instead of with its denominator.

Solution by Professor SEBASTIAN SIRCOM, M.A.

1. The continued fraction may be written

$$1+\frac{1}{1+}\frac{1.2}{1+}\frac{2.3}{1+}\frac{3.4}{1+}\ldots,$$

the numerators and denominators of the successive convergents to which are respectively connected by the relation $u_n = u_{n-1}+(n-1)(n-2)\,u_{n-2}$, the solution of which is cP_n+kQ_n, where P_n, Q_n are the products of n terms of the progressions 1, 1, 3, 3, 5, 5, ..., 1, 2, 2, 4, 4 ... (Quest. 2845, Vol. XIII., p. 50). Hence, when n is infinite, the value of the continued fraction is

$$\frac{1\,.\,2^2\,.\,4^2\ldots}{1^2\,.\,3^2\,.\,5^2\ldots}=\tfrac{1}{2}\pi.$$

2. In both cases the successive convergents will be alternately too small and too great; but, leaving off with a numerator 1, we have for the last denominator in the $(n+1)^{\text{th}}$ convergent $\frac{1}{n-1}+1$, and leaving off with its denominator this will be replaced by $\frac{1}{n-1}+n$, so that, if $\frac{1}{n-1}+1$ gives a convergent larger or smaller than the true value, $\frac{1}{n-1}+n$ will make it still larger or smaller, and consequently will not give so near an approximation.

8825. (Professor GENESE, M.A.)—If S, S′ be isogonal points of a triangle ABC (*i.e.*, foci of an in-conic), the conic ABCSS′ touches each of the sixteen conics which can be drawn through S, S′ to touch the sides of the four triangles determined by the bisectors of the angles A, B, C, taking one at a time through each vertex.

Solution by Professor NASH, M.A.

Let A′B′C′ be the triangle formed by the external bisectors, L the intersection of the internal bisectors of the angles.

The lines AA′, AB′ bisect the interior and exterior angles between AB, AC, and also those between AS, AS′; therefore, if the figure be so projected that S, S′ become the circular points, I, J, AA′, AC′ are harmonic conjugates of AI, AJ, and therefore are at right angles; therefore, in projected as well as in original figure, ALA′, BLB′, CLC′ are the perpendiculars of the triangle A′B′C′.

The conic ABCSS′ projects into the nine-point circle of A′B′C′, which is also the nine-point circle of the triangles LB′C′, LC′A′, LA′B′, and this touches the sixteen in- and ex-circles of the four triangles, which are the projections of the sixteen conics named in the question.

9280. (Major-General P. O'Connell.)—A reservoir full of water is of the form of an inverted cone. It is required to find an expression for the time in which its surface will fall through a given depth, if water is being drawn off at a uniform rate through sluices, *i.e.*, the same volume in each successive second of time, while a uniform depth per second is being removed by evaporation.

Solution by D. Biddle.

Let r = radius of base of cone, the height being unity; m = depth of fluid *evaporated* in one second; $n\pi r^2$ = amount of fluid *drawn off* by sluices in one second (that is to say, n = depth which such fluid would occupy in a cylindrical reservoir of equal base to the cone's); a, b = distances of surface of fluid from apex of cone, at commencement and finish of flow; and x = varying distance during the flow. Then $m + n/x^2$ = rate of depression per second at x,

and t, the time required, $= \int_b^a dx \Big/ \left(m + \frac{n}{x^2}\right) = \int_b^a \frac{x^2 dx}{n + mx^2}$

$$= \frac{a-b}{m} - \frac{1}{m}\left(\frac{n}{m}\right)\left\{\tan^{-1} a\left(\frac{m}{n}\right)^{\frac{1}{2}} - \tan^{-1} b\left(\frac{m}{n}\right)^{\frac{1}{2}}\right\}.$$

Thus, if $a = \cdot 8$, $b = \cdot 3$, $m = \cdot 000001$, and $n = \cdot 001$,

$$t = 500{,}000 - 1{,}000{,}000\,(1000)^{\frac{1}{2}}\left\{\tan^{-1}\left(\tfrac{64}{100000}\right)^{\frac{1}{2}} - \tan^{-1}\left(\tfrac{9}{100000}\right)^{\frac{1}{2}}\right\}$$
$$= 500{,}000 - 1{,}000{,}000\,(1000)^{\frac{1}{2}}\left\{1^\circ\,26'\cdot 9502 - 32'\cdot 6122\right\}$$
$$= 161\cdot 6 \text{ seconds.}$$

9238. (Professor B. Hanumanta Rau, M.A.)—If AA′, BB′, CC′ are equal tangents drawn in the same sense to the circum-circle of a triangle ABC; prove that A′B′ : B′C′ : C′A′ = AB : BC : CA.

Solution by Professor De Wachter; R. F. Davis, M.A.; *and others.*

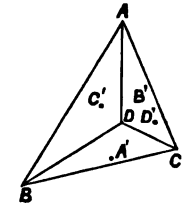
The rectangular triangles OAA′, OBB′ and OCC′ are equal to one another; therefore

(1) OA′ = OB′ = OC′; (2) $\widehat{AOB} = \widehat{A'OB'}$, $\widehat{BOC} = \widehat{B'OC'}$, $\widehat{COA} = \widehat{C'OA'}$.

Hence the triangle A′B′C′ is similar to ABC, &c.

9262. (Professor NEUBERG.)—Soit O′ le conjugué isogonal (ou l'inverse) du centre de la sphère circonscrite à un tétraèdre ABCD. Démontrer que ce point est le centre de la sphère inscrite au tétraèdre, qui a pour sommets ses projections sur les faces du tétraèdre ABCD.

Solution by SAMUEL ROBERTS, M.A.



Let O, O′ be respectively the centre of the sphere circumscribing the tetrahedron ABCD, and the isogonal conjugate, and let A′, B′, C′, D′ be the orthogonal projections of O′ on the faces opposite A, B, C, D, respectively. Then the tetrahedron A′B′C′D′ is similar and homothetic to the tetrahedron formed by the tangent planes at A, B, C, D of the sphere circumscribing ABCD. For, by a known theorem, AO, BO, CO, DO are perpendicular to corresponding faces of A′B′C′D′. Also, if A″, B″, C″, D″ are the projections of O on the corresponding faces of ABCD, we have, by hypothesis,

$$O'A' \cdot OA'' = O'B' \cdot OB'' = O'C' \cdot OC'' = O'D' \cdot OD'' = k^2.$$

But the angles between the lines OD″, OA, and between O′D′, O′P′ (O′P′ denoting the distance of O′ from the plane D′B′C′) are equal, because the corresponding constituents of these pairs of lines are perpendicular to the same planes. Hence, we have

$$OD'' = OA\cos\phi,\ O'D'\cos\phi = O'P',\ \text{.or finally } O'P' = \frac{k^2}{OA} = \frac{k^2}{R}.$$

This value being independent of the particular face passing through D′, O′ is the centre of the inscribed tetrahedron.

9254. (W. J. C. SHARP, M.A.)—If any point P in its plane be joined to the vertices of a triangle ABC, and if R_1, R_2, R_3 be the radii of the circles described about BPC, CPA, APB, respectively; then, a, b, c being the sides of the triangle,

$$\frac{a^2 \cdot b^2 \cdot c^2}{R_1^2 \cdot R_2^2 \cdot R_3^2} = \left(\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3}\right)\left(-\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3}\right)$$
$$\left(\frac{a}{R_1} - \frac{b}{R_2} + \frac{c}{R_3}\right)\left(\frac{a}{R_1} + \frac{b}{R_2} - \frac{c}{R_3}\right).$$

Solution by G. G. STORR, M.A.; W. J. GREENSTREET, M.A; *and others.*

If $A+B+C = \pi$, then $(\pi-A)+(\pi-B)+(\pi-C) = 2\pi$; so that the relations subsisting between the sines of three angles whose sum $= 2\pi$ must be the same as that between the angles of a triangle.

Now, in any triangle,

$$(a+b+c)(-a+b+c)(a-b+c)(a+b-c) = 16S^2, \text{ where } S = \text{area}.$$

$$\therefore (\sin A+\sin B+\sin C)(-\sin A+\sin B+\sin C)(\sin A-\sin B+\sin C)$$
$$\times(\sin A+\sin B-\sin C) = S^2/R^4 = 4\sin^2 A\sin^2 B\sin^2 C \ldots\ldots(A),$$

for $S = \frac{1}{2}bc \sin A = 2R^2 \sin A \sin B \sin C$.

In the given example $a/2R_1 = \sin BPC$, and $BPC + CPA + APB = 2\pi$. Hence a relation (B) similar to (A) subsists between the sines of these angles; and substituting in (B), we have

$$\left(\frac{a}{2R_1} + \frac{b}{2R_2} + \frac{c}{2R_3}\right)(- + +)(+ - +)(+ + -) = 4\,\frac{a^2}{4R_1^2} \cdot \frac{b^2}{4R_2^2} \cdot \frac{c^2}{4R_3^2},$$

which agrees with the result in the question.

[*Otherwise*:—Since $(\sin X + \sin Y + \sin Z)(\sin Y + \sin Z - \sin X) \ldots$
$= (4 \sin \frac{1}{2}X \sin \frac{1}{2}Y \sin \frac{1}{2}Z)(4 \cos \frac{1}{2}Y \cos \frac{1}{2}Z \sin \frac{1}{2}X) \ldots$
$2n$ or 0, so that the case when p is external to the triangle is included,
when $X + Y + Z = 2\pi$, $\quad = 4 \sin^2 X \sin^2 Y \sin^2 Z$,
denoting the angles at P by α, β, γ, we get $R_1 = a/2 \sin \alpha$, etc.;
therefore $\quad L \,.\, H = 64 \sin^2 \alpha \sin^2 \beta \sin^2 \gamma$,
$R \,.\, H = 16 (\sin \alpha + \sin \beta + \sin \gamma)(\sin \beta + \sin \gamma - \sin \alpha) \ldots$
$= (\text{by Lemma}) \; 64 \sin^2 \alpha \sin^2 \beta \sin^2 \gamma$.

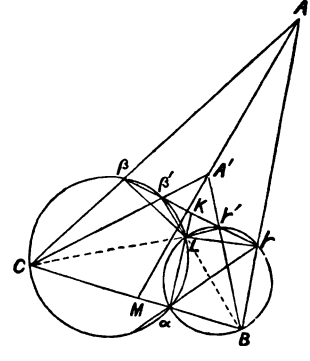
9206. (Professor SCHOUTE.)—Given the triangle ABC; prove that there are three tangents A'BC, AB'C, ABC' having in common with ABC two vertices and the Lemoine point; and that the new triangle A'B'C' is in perspective with ABC.

Solution by H. PLAMENEWSKY, M.A.; Professor NASH, M.A.; *and others.*

Abaissons du point de Lemoine L perpendiculaires $L\alpha$, $L\beta$, $L\gamma$ sur les côtés du triangle ABC.

Pour le triangle $\alpha\beta\gamma$, L est centre de gravité; $L\alpha$ divise $\beta\gamma$ en deux parties égales $K\beta = K\gamma$; $\beta\gamma$ est perpendiculaire à la médiane AM.

Ecrivons deux cercles passants par les points $\alpha L\beta C$ et par $\alpha L\gamma B$. Soient β' et γ' deux autres points où $\beta\gamma$ coupe ces deux cercles. Il est évident que $\beta\beta' = \gamma\gamma'$.



En effet,

$K\beta \,.\, K\beta' = K\alpha \,.\, LK = K\gamma \,.\, K\gamma'$;

mais $K\beta = K\gamma$, d'où $K\beta' = K\gamma'$.

Si A' et A_2 sont les points d'intersection $C\beta'$ et $B\gamma'$ avec la médiane AM, nous avons;

$$\frac{AC}{L\beta} = \frac{A_1C}{L\beta'} = \frac{AA'}{\beta\beta'} \text{ et } \frac{AB}{L\gamma} = \frac{A_2B}{L\gamma_1} = \frac{AA_2}{\gamma\gamma_1};$$

mais $\dfrac{BC}{L\alpha} = \dfrac{AC}{L\beta} = \dfrac{AB}{L\gamma}$, d'où $\dfrac{AA_1}{\beta\beta_1} = \dfrac{AA_2}{\gamma\gamma_1}$, ou $AA_1 = AA_2$ .....(1),

donc A_1 se confond avec A_2, $\frac{BC}{L\alpha} = \frac{A'C}{L\beta_1} = \frac{B'A}{L\gamma_1}$ .....(2),

donc L est aussi le point de Lemoine du triangle A'BC ..... (3),

A' se trouve sur la médiane AM, qui est commune pour deux triangles. ABC et A'BC. Ainsi B' et C' se trouvent sur deux autres médianes, et par conséquent les triangles ABC et A'B'C' sont homologiques et le centre de gravité du triangle ABC est centre d'homologie.

8750. (Professor MATHEWS, M.A.)—Denoting by $[x]$ the greatest integer contained in the numerical quantity x, prove that, if a, b are positive integers prime to each other,

$$\left\{\frac{b}{a}\right\} + \left\{\frac{2b}{a}\right\} + \left\{\frac{3b}{a}\right\} + \ldots \left\{\frac{(a-1)\,b}{a}\right\}$$
$$= \left\{\frac{a}{b}\right\} + \left\{\frac{2a}{b}\right\} + \left\{\frac{3a}{b}\right\} + \ldots \left\{\frac{(b-1)\,a}{b}\right\} = \tfrac{1}{2}(a-1)(b-1).$$

Solution by Professor BEYENS; KATE GALE; *and others.*

Calling the two expressions A and B, we have

$$\frac{b}{a} + \frac{2b}{a} + \ldots \frac{(a-1)\,b}{a} = A + \text{remainders} = A + \frac{1+2+\ldots(a-1)}{a}.$$

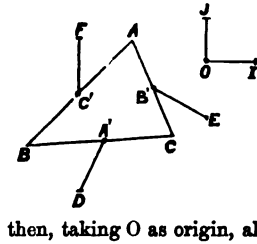
Therefore $\frac{b}{a} \cdot \frac{(a-1)\,a}{2} - \frac{(a-1)\,a}{2a} = A = \frac{(b-1)(a-1)}{2}$;

and B similarly.

9218. (A. GORDON.)—Show that, if lines are drawn perpendicular to and through the middle points of the sides of a scalene triangle, outwards, and distances measured along these lines proportional to the corresponding sides, there is only one set of points so found which can form an equilateral triangle.

Solution by J. W. SHARPE, M.A.

Let ABC be the triangle, A', B', C' middle points of the sides, A'D, B'E, C'F drawn making the same angles with, and proportional to, the sides BC, CA, AB respectively.



We shall prove that, when DEF is equilateral, A'D, &c. must be perpendicular to BC, &c., and

$$A'D : A'B = 1 : \sqrt{3}.$$

Take two straight lines OI, OJ, at right angles to each other, and of unit length; then, taking O as origin, all

vectors in the plane may be treated as numerators of co-planar quaternions (clinants, as they have been called) with common denominator OI; and they may be denoted by the small letters corresponding to the capital letters in the figure, *i.e.*,

$$\frac{OA}{OI} = \frac{a}{i} = a; \quad \frac{OJ}{OI} = \frac{j}{i} = j = i(-1)^{\frac{1}{2}};$$

thus all the small letters are symbols of clinants, and follow the laws of ordinary algebra. Now let

$$k = \frac{A'D}{A'B} = \frac{B'E}{B'C} = \frac{C'F}{C'A}; \text{ therefore } \frac{A'D}{A'B} = \frac{d - \frac{1}{2}(b+c)}{\frac{1}{2}(b-c)} = k;$$

therefore $2d = b(1+k) + c(1-k)$, $2e = c(1+k) + a(1-k)$,

$$2f = a(1+k) + b(1-k).$$

Now, if DEF be equilateral, then must

$$\frac{f-e}{d-e} = \frac{f-d}{f-e}; \quad i.e., \ (d-e)(f-d) = (f-e)^2.$$

Substituting for d, e, f, and reducing, the result is

$$3k^2 + 1 = 0; \text{ therefore } k = \pm j \times \frac{1}{\sqrt{3}};$$

therefore $\angle$ BA′D must be a right angle, and

$$\text{length A'D} : \text{length A'B} = 1 : \sqrt{3};$$

and in consequence of the double sign of k, a second equilateral triangle can be drawn by producing DA′ inwards to D′, where D′A′ = A′D, &c. &c.

[*Otherwise*:—Let $aa' = \lambda(BC) = \lambda a$, $bb' = \lambda b$, &c.

$(a'b')^2 = a^2(\frac{1}{4} + \lambda^2) + b^2(\frac{1}{4} + \lambda^2) - 2ab(\frac{1}{4} + \lambda^2)\cos(C + 2\theta)$,

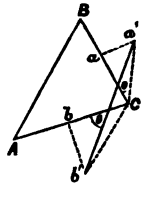
$\left(\text{where } \tan\theta = 2\lambda, \text{ therefore } \sin\theta = \frac{2\lambda}{(1+4\lambda^2)^{\frac{1}{2}}}, \text{ \&c.}\right)$;

therefore

$(a'b')^2 = (a^2 + b^2)(\lambda^2 + \frac{1}{4}) - \frac{1}{4}(2ab\cos C)(1 - 4\lambda^2) + 2ab\sin C\,\lambda.$

Hence $(a'b')^2 - (b'c')^2 = (a^2 - c^2)\left\{\lambda^2 + \frac{1}{4} + 2(\lambda^2 - \frac{1}{4})\right\} = 0$

(substituting for $2ab\cos C$ its value $a^2 + b^2 - c^2$, &c.)



Therefore $3\lambda^2 = \frac{1}{4}$, therefore $2\lambda = \frac{1}{\sqrt{3}} = \tan\theta$, therefore $\theta = 30°$.

7341. (A. Martin, B.A.)—Solve the equations

$$yz(y+z-x) = a, \quad xz(z+x-y) = b, \quad xy(x+y-z) = c.$$

Solution by Professor Lampe, LL.D.

The solution given by the Rev. T. C. Simmons (p. 29 of Vol. xlii.; see also p. 26 of Vol. xl.) is remarkably elegant and quite satisfactory, but there is still the last step to be done. Putting $x/z = p$, the cubic to

be solved is (1) $4p^3-(bc+ca+ab)\,p-abc=0$. Calculating the values of x^3, y^3, z^3, we obtain (2^a) $x=\dfrac{W}{2p+a}$, $y=\dfrac{W}{2p+b}$, $z=\dfrac{W}{2p+c}$, where $W=\sqrt[3]{p\,(2p+a)(2p+b)(2p+c)}$. These expressions are most convenient for numerical calculation, and show which values of x, y, z correspond to one value of p; they admit however of further simplification.

We have $x^3=\dfrac{p\,(2p+b)(2p+c)}{(2p+a)^2}$. Substituting in the expansion of the numerator the value of $4p^3$ taken from (1), the numerator proves to be divisible by $2p+a$, and x^3 becomes (2^b) $x^3=\dfrac{p\,(b+c)+bc}{2p+a}$; similarly y^3 and z^3.

This simplified expression shows how to find the cubic equation for x^3. Solving for p and substituting its value in (1), we get

$$Ax^9+Bx^6+Cx^3+D=0,$$

where $A=4a\,(a-b)(a-c)$,

$B=4\left\{(bc+ca+ab)^2-3\,(a+b+c)\,abc\right\}$

$\quad=4\left\{b^2c^2+c^2a^2+a^2b^2-a^2bc-ab^2c-abc^2\right\}$,

$C=12ab^2c^2-(bc+ca+ab)\,(4bc+ca+ab)(b+c)+6\,(b+c)^2\,abc$

$\quad=-a^2\,(b+c)^3+abc\,(b^2+c^2+14bc)-4b^2c^2\,(b+c)$,

$D=b^3c^3\,(b-c)^2$.

Besides the formula (2^b) we have

$$(2^c)\quad x^3=\tfrac{1}{2}\left(b+c+\frac{2bc-ab-ac}{2p+a}\right);$$

therefore we conclude that x is independent of p, if $2bc=ab+ac$. In this case the cubic for x^3 becomes a perfect cube, and x^3 has three equal real values $=\frac{1}{2}\,(b+c)$, while those of y and z must be calculated from p.

[We add two examples to illustrate the foregoing considerations:—

(1) $\quad a=60,\ b=24,\ c=6,\ p^3-486p-2160=0.$

$p_1=24,\ p_2=-12+3\sqrt{6},\ p_3=-12-3\sqrt{6}.$

$x_1=2,\ x_2=\sqrt[3]{-10\cdot2+4\cdot2\sqrt{6}},\quad x_3=\sqrt[3]{-10\cdot2-4\cdot2\sqrt{6}}.$

$y_1=3,\ y_2=\sqrt[3]{33-12\sqrt{6}},\quad y_3=\sqrt[3]{33+12\sqrt{6}}.$

$z_1=4,\ z_2=\sqrt[3]{-156-66\sqrt{6}},\quad z_3=\sqrt{-156+66\sqrt{6}}.$

(2) $\quad a=3,\ b=2,\ c=6,\ p^3-9p-9=0\ \ (2bc=ab+ac).$

$p_1=3{\cdot}411474\ (5),\ p_2=-2{\cdot}226681\ (7),\ p_3=-1{\cdot}184792\ (7).$

$x_1=\sqrt[3]{4},\quad x_2=\sqrt[3]{4},\quad x_3=\sqrt[3]{4}=1{\cdot}587401.$

$y_1=1{\cdot}767318,\quad y_2=0{\cdot}940370\ (5),\quad y_3=-2{\cdot}707688.$

$z_1=1{\cdot}216019\ (7),\ z_2=-1{\cdot}491669\ (1),\ z_3=0{\cdot}2756493\ (5).$

The above form also shows the correspondence of their impossible values.]

9246. (Professor Bordage.)—Solve the equation

$$16^{\cos^2 x+2\sin^2 x}+4^{2\cos^2 x}=40.$$

Solution by Prof. BENI M. SARKAR; Major-Gen. O'CONNELL; *and others.*

The equation reduces to $4^{4-2\cos^2 x} + 4^{2\cos^2 x} = 40$,

or, putting y for $4^{2\cos^2 x}$, $\qquad y + \frac{256}{y} = 40$,

whence $y = 32$ or 8, *i.e.*, $2\cos^2 x = \frac{5}{2}$ or $\frac{3}{2}$, and, taking the only admissible value $\frac{3}{4}$, $\cos^2 x = \frac{3}{4}$, that is, $x = \frac{1}{6}\pi$, or generally $2n\pi \pm \frac{1}{6}\pi$.

8098. (W. J. C. SHARP, M.A.)—Show that, if p be any prime and n an integer prime to p, $x^{n(p-1)} + x^{n(p-2)} + x^{n(p-3)} + \ldots + 1$ is divisible by $x^{p-1} + x^{p-2} + x^{p-3} + \ldots + 1$; and, if n be even, also by

$$x^{p-1} - x^{p-2} + x^{p-3} + \ldots + (-1)^{p-1};$$

and that, when n is odd, this last quantity divides

$$x^{n(p-1)} - x^{n(p-2)} + x^{n(p-3)} + \ldots + (-1)^{p-1}.$$

Solution by Professor SEBASTIAN SIRCOM, M.A.

We have $\qquad x^p - 1 \equiv (x-1)(x^{p-1} + x^{p-2} + \ldots + 1)$,

and $\qquad x^{np} - 1 \equiv (x^n - 1)(x^{n(p-1)} + x^{n(p-2)} + \ldots + 1)$ .....(1),

also $\qquad x^{np} - 1 \equiv (x^p - 1)(x^{p(n-1)} + \ldots + 1)$ .....(2).

Hence, since $x^{p-1} + x^{p-2} + \ldots + 1$ divides (2) without remainder, and has no factor in common with $x^n - 1$, it must divide $x^{n(p-1)} + x^{n(p-2)} + \ldots + 1$.

[Since $\qquad f[\phi'(x) + \mathrm{D}]\, u = e^{-\phi(x)} f(\mathrm{D})(e^{\phi(x)} u)$,

we have $\qquad e^{\frac{1}{2}a[\phi'(x)+\mathrm{D}]^2} e^{\frac{1}{2}b\mathrm{D}^2} = e^{-\phi(x)}\, e^{\frac{1}{2}a\mathrm{D}^2} e^{\phi(x)} e^{\frac{1}{2}b\mathrm{D}^2}$.

Ex.—Let $\phi(x) = \log x$, then

$$e^{\frac{1}{2}a(1/x+\mathrm{D})^2} e^{\frac{1}{2}b\mathrm{D}^2} = (1-ab)^{-\frac{1}{2}} e^{b/(1-ab)\,\frac{1}{2}x^2}.]$$

If n be even $= 2m$, $\qquad x^{2mp} - 1 \equiv (x^{2p} - 1)(x^{2p(m-1)} + \ldots + 1)$,

and $x^{2p} - 1$ is divisible by $x^{p-1} - x^{p-2} + \ldots + (-1)^{p-1}$,

therefore, as before, $x^{2m(p-1)} + \ldots + 1$ is so divisible. If n be odd $= 2m+1$,

$$x^{(2m+1)p} + 1 \equiv (x^{2m+1} + 1)(x^{(2m+1)(p-1)} - x^{(2m+1)(p-2)} + \ldots + 1)$$
$$\equiv (x^p + 1)(x^{2mp} - \ldots + 1).$$

Since p is prime, $x^p + 1$ is divisible by $x^{p-1} - x^{p-2} + \ldots (-1)^{p-1}$; therefore, as before, $x^{n(p-1)} - x^{n(p-2)} + \ldots (-1)^{p-1}$ is so divisible if n is odd.

9209. (Professor MAHENDRA NATH RAY, M.A., LL.B.)—Show that the value of $\dfrac{6+4x}{3x^2} - \dfrac{1}{x - \log_e(1+x)}$, when $x = 0$, is $\frac{1}{5}$.

Solution by R. Holmes, B.A. ; D. Watson, M.A. ; *and others.*

$$\frac{6+4x}{3x^2}-\frac{1}{x-\log_e(1+x)}=\frac{6+4x}{3x^2}-\frac{1}{\frac{1}{2}x^2-\frac{1}{3}x^3+\frac{1}{4}x^4-\ldots}$$

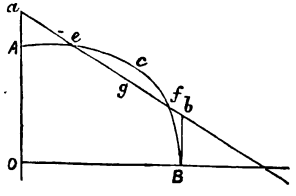
$$=\frac{3x^2-2x^3+\frac{3}{2}x^4\ldots+2x^3-\frac{4}{3}x^4+\ldots-3x^2}{\frac{3}{2}x^4-x^5+\ldots}=\frac{\frac{1}{6}x^4}{\frac{3}{2}x^4}\text{ (when } x=0)=\frac{1}{9}.$$

8885. (Rev. T. W. Openshaw, M.A.) — Through the cosinusoid A*e*C*f* B having the abscissa OB = $\frac{1}{2}\pi$, and the extreme ordinate

$$OA=\cos 0=1,\quad OB=\cos\tfrac{1}{2}\pi=0,$$

draw a straight line *aefb* so that the area A*ae* + *egf* C + *fb*B shall be a minimum.

Solution by the Proposer.



Let the straight and cosinusoid lines be

$$y=mx+c \text{ and } y=\cos x.$$

Combining, $mx+c=1-\frac{1}{2}x^2$ approximately; or $x^2+2mx+2(c-1)=0$, if x_1, x_2 be the roots, $x_1+x_2=-2m$, $x_1x_2=2(c-1)$. Area of the spaces required = u,

$$=\int_{x_1}^{0}(y_1-y_2)\,dx+\int_{x_2}^{x_1}(y_2-y_1)\,dx+\int_{\frac{1}{2}\pi}^{x_2}(y_1-y_2)\,dx$$

$$=\left[\frac{mx^2}{2}+cx-\sin x\right]_{x_1}^{0}+\left[\ \right]_{x_1}^{x_2}+\left[\ \right]_{\frac{1}{2}\pi}^{x_2}$$

$$=m(x_2^2-x_1^2)+2c(x_2-x_1)-2(\sin x_2-\sin x_1)-\tfrac{1}{8}m\pi^2-\tfrac{1}{2}c\pi+1,$$

$$\frac{du}{dx_1}=\frac{3x_1^2+2x_1x_2-x_2^2}{2}+x_2^2-2x_1x_2-2+2\cos x_1+\frac{\pi^2}{16}-\frac{\pi x_2}{4}=0,$$

(substituting for m &c., and differentiating),

$$\frac{du}{dx_2}=\frac{x_1^2-2x_1x_2-3x_2^2}{2}-x_1^2+2x_1x_2+2-2\cos x_2+\frac{\pi^2}{16}-\frac{\pi x_1}{4}=0;$$

whence, taking $\cos x=1-\frac{1}{2}x^2$,

$$x_1^2-2x_1x_2+x_2^2-\tfrac{1}{2}\pi x_2+\tfrac{1}{8}\pi^2=0,$$

and

$$x_1^2-2x_1x_2+x_2^2+\tfrac{1}{2}\pi x_1-\tfrac{1}{8}\pi^2=0,$$

therefore $x_1+x_2=\frac{1}{2}\pi$, therefore $m=-\frac{1}{4}\pi$,

$$2(x_1-x_2)^2+\tfrac{1}{2}\pi(x_1-x_2)=0,\text{ therefore } x_1-x_2=0 \text{ or } -\tfrac{1}{4}\pi,$$

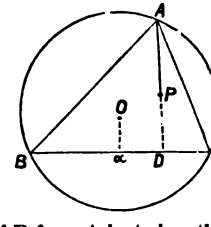
therefore $x_1x_2=\frac{1}{16}\pi^2$ or $\frac{1}{16}(\pi^2+\pi)$;

therefore lines are $y=-\frac{1}{4}\pi x+\frac{1}{32}\pi^2+1$, a tangent,

or $y=-\frac{1}{4}\pi x+\frac{1}{32}(\pi^2+\pi)+1.$

9214. (Rev. T. C. Simmons, M.A. Connected with Quest. 9099.)—Show how to construct a system of triangles having one given point for circumcentre and another given point for orthocentre, and having given either the product of the cosines of the angles (not $> \frac{1}{8}$), or the sum of the squares of the sides. Show, moreover, that the envelope of the sides of any such system of triangles is an ellipse having the two given points for foci.

Solution by R. Holmes, B.A.; Professor Sarkar, M.A.; *and others.*



If O be the circumcentre, and P the orthocentre, then $OP^2 = R^2(1 - 8\cos A\cos B\cos C)$, and $a^2 + b^2 + c^2 = 4R^2(\sin^2 A + \sin^2 B + \sin^2 C)$

$$= 8R^2(1 + \cos A\cos B\cos C).$$

Hence, if either $\cos A\cos B\cos C$ or $a^2 + b^2 + c^2$ be given, R the radius of circumcircle is known.

Let ABC be circumcircle, through P draw any straight line PA cutting circumcircle in A, draw $O\alpha$ parallel and equal to $\frac{1}{2}PA$, and through α perpendicular to $O\alpha$ draw the chord BC; then ABC has P for orthocentre, since the distance of P from A is twice the distance of O from BC.

Again, $O\alpha$ varies as cos A, and PD the trilinear coordinate of P varies as $\cos B\cos C$; thus $O\alpha \cdot PD$ varies as $\cos A\cos B\cos C$ and is given; thus the envelope of the sides is an ellipse foci O and P.

9244. (Professor Munindranath Bhattacharyya, M.A.)—Professor Wolstenholme obtains very elegantly by parts the integrals of $e^{mx}\cos nx$ and $e^{mx}\sin nx$. Show that the integrals can be obtained by a different process. [Gregory, in his *Methods of Diff. and Integ. Calc.*, suggests that these two integrals are obtainable by "Double Integration by Parts."]

Solution by W. J. C. Sharp, M.A.; *the* Proposer; *and others.*

$$\int e^{mx}\cos nx\,dx = \tfrac{1}{2}\int e^{mx}(e^{inx} + e^{-inx})\,dx$$

$$= \tfrac{1}{2}\left[\int e^{x(m+in)}\,dx + \int e^{x(m-in}\,dx\right] = \tfrac{1}{2}\left[\frac{e^{x(m+in)}}{m+in} + \frac{e}{m-in}\right]$$

$$= \tfrac{1}{2}e^{mx}\left[\frac{m(e^{inx} + e^{-inx}) - in(e^{inx} - e^{-inx})}{m^2 + n^2}\right]$$

$$= e^{mx}\left[\frac{m\cos nx + n\sin nx}{m^2 + n^2}\right].$$

Similarly, $\int e^{mx}\sin nx\,dx = e^{mx}\left[\frac{m\sin nx - n\cos nx}{m^2 + n^2}\right]$.

9103. (Asparagus.)—Prove that the envelope of the hyperbolas

$$\frac{x^2}{2-\sin\theta}-\frac{y^2}{2+\sin\theta}=a^2\sin\theta$$

(θ being the parameter) is a four-cusped hypocycloid.

Solution by Professor Wolstenholme, M.A., Sc.D.

For points on the envelope of the hyperbolas

$$\frac{x^2}{2-\sin\theta}-\frac{y^2}{2+\sin\theta}=a^2\sin\theta,$$

we have, by the usual process, $\dfrac{x^2}{(2-\sin\theta)^2}+\dfrac{y^2}{(2+\sin\theta)^2}=a^2$;

whence $$\frac{x^2(1-\sin\theta)}{(2-\sin\theta)^2}=\frac{y^2(1+\sin\theta)}{(2+\sin\theta)^2},$$

or $$\frac{x^2}{(1+\sin\theta)(2-\sin\theta)^2}=\frac{y^2}{(1-\sin\theta)(2+\sin\theta)^2}$$

$$=\frac{\dfrac{x^2}{2-\sin\theta}-\dfrac{y^2}{2+\sin\theta}}{(1+\sin\theta)(2-\sin\theta)-(1-\sin\theta)(2+\sin\theta)}=\frac{a^2\sin\theta}{2\sin\theta},$$

whence $x=\pm\sqrt{2}\,a\,(\cos^3\frac{1}{2}\theta-\sin^3\frac{1}{2}\theta)$, $y=\pm\sqrt{2}\,a\,(\cos^3\frac{1}{2}\theta+\sin^3\frac{1}{2}\theta)$,

the ambiguities being independent, and the envelope is the four-cusped epicycloid $$(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=2a^{\frac{2}{3}}.$$

Each hyperbola touches the envelope in four real points

$$x^2=\tfrac{1}{8}a^2(1+\sin\theta)(2-\sin\theta)^2,\quad y^2=\tfrac{1}{8}a^2(1-\sin\theta)(2+\sin\theta)^2,$$

and cuts it in four other real points

$$x^2=\tfrac{1}{16}a^2(2-\sin\theta)(4+\sin\theta)^2,\quad y^2=\tfrac{1}{16}a^2(2+\sin\theta)(4-\sin\theta)^2.$$

All the points on the envelope will be obtained by taking θ from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$. The curvatures of the hyperbola and the envelope at any point of contact are as $3\sin^2\theta : 4-\sin^2\theta$; any branch of the epicycloid with which the hyperbola has double contact will lie altogether within the hyperbola, and when $\theta=\pm\frac{1}{2}\pi$, the hyperbola osculates its envelope.

8341. (J. Brill, M.A.)—ABCDEF is a hexagon inscribed in a circle. Prove that

$$\triangle BCD\,.\,\triangle EFA-\triangle CDA\,.\,\triangle EFB+\triangle DAB\,.\,\triangle EFC-\triangle ABC\,.\,\triangle EFD=0.$$

Solution by A. Gordon; Professor Nash, M.A.; *and others.*

Let A, B, C, D, E, F be the points taken in order, then, by a known proposition (Townsend),

$$\triangle BCD\,.\,(EA)^2-\triangle CDA\,.\,(EB)^2+DAB\,.\,(EC)^2-ABC\,.\,(ED)^2=0\ldots(1),$$

and $BCD \cdot (FA)^2 - CDA \cdot (FB)^2 + DAB \cdot (FC)^2 - ABC \cdot (FD)^2 = 0 \ldots (2)$.

But $(FA)^2 + (FE)^2 = (FE)^2 + 2AE \cdot AF \sin A \cot A$ (A is constant).

Adding (1) and (2), and substituting for $(FA)^2 + (EA)^2$, &c., we have the required equation.

9127. (B. Hanumanta Rau, M.A.)—Find a point in a given straight line such that the tangents from it to two given circles may be equally inclined to the given straight line.

Solution by Professors E. H. Culley, M.A.; Ignacio Beyens; *and others.*

Let C and C′ be the two circles. L the line. Take the reflection C″ of either circle with regard to the line L, and draw the four common tangents of C″ and the remaining circle. These tangents will cut L in the required points.

9275. (Captain H. Brocard.)—Les pieds des hauteurs d'un triangle sont les foyers de paraboles tangentes à deux côtés et aux deux hauteurs correspondantes. Ces paraboles admettent pour directrices, pour paramètres et pour sommets, respectivement, les côtés et les hauteurs du triangle orthocentrique et les milieux des hauteurs de ce triangle. Donner l'expression analytique des paramètres de ces paraboles.

Solution by R. F. Davis, M.A.

If ABC be the triangle; P, Q, R the feet of its perpendiculars, then latus rectum of one of the parabolas = twice perpendicular from P on QR

$$= 2PQ \sin PQR = 2c \cos C \sin (\pi - 2B) = 2R \sin 2B \sin 2C.$$

8693. (R. Tucker, M.A.)—If P, Q, R are three points on a parabola, such that $\Sigma y^{-1} = 0$, then the centres of curvature at these points are collinear. If they are three points which have a co-normal point (O), then the radical centre (C) of the circles of curvature will lie on the directrix, if the locus of O is a certain straight line parallel to the directrix. If the locus of O is a certain hyperbola, the polar of C with regard to the parabola passes through O; and if the locus is a certain semicubical parabola, C will lie on the parabola.

Solution by R. Knowles, B.A.; Belle Easton; *and others.*

The coordinates of the centre of curvature at P, Q, R are $2a + 3x_1$, $-3y_1^3/4a^2$, &c.; the lines joining the centres of curvature at P, Q and

P, R are inclined to the axis at angles

$$= \tan^{-1}(y_1^2 + y_1y_2 + y_2^2)/(y_1 + y_2) \quad \ldots\ldots(1),$$

and $$\tan^{-1}(y_1^2 + y_1y_3 + y_3^2)/(y_1 + y_3) \quad \ldots\ldots(2);$$

but, from the condition $\Sigma y^{-1} = 0$, we have $y_3 = -\frac{y_1y_2}{y_1+y_2}$; making this substitution, (1), (2) are identical, therefore the centres are collinear.

Since the normals at P, Q, R meet in O (X, Y), we have

$$y_1 + y_2 + y_3 = 0, \quad 4a(X-2a) = y_1^2 + y_1y_2 + y_2^2, \quad 8a^2Y = y_1y_2y_3,$$

and we readily find the coordinates of C to be $\frac{2a-X}{2}$, $\frac{3aY}{4(X-2a)}$. If C is on the directrix, we have $X = 4a$, therefore locus of O is a line parallel to the directrix. The equation to the polar of C with respect to the parabola is $\frac{3aY}{4(X-2a)}y = 2a\left(x + \frac{2a-X}{2}\right)$, and, if this passes through O, we have for the locus of O the hyperbola $3Y^2 - 4X^2 + 16a^2 = 0$. If C is on the parabola, the locus of O is the semi-cubical parabola

$$9aY^2 = 32(2a-X)^3.$$

9232. (Professor Hudson, M.A.)—If the bisector of an angle of a triangle pass through the centre of the nine-points circle, prove that that angle must be 60°.

Solution by R. F. Davis, M.A.; J. O'Byrne Croke, M.A.; *and others.*

Let ABC be a triangle; P, Q, R the feet of its perpendiculars, and D, E, F the mid-points of its sides. Then, if the centre of the nine-points circle lie on the bisector of the angle A, this circle intersects CA, AB symmetrically; the distances from A of the two points of intersection with one side being respectively equal to the distances of the points of intersection with the other.

This can happen (1) when AE = AF and AQ = AR; so that the property holds for any isosceles triangle.

(2) When AQ = AF = ½AB and AR = AE = ½AC; and in this case the angle A must obviously be 60°.

[*Otherwise*:—Substituting the coordinates of the nine-point centre, K cos (B−C), K cos (C−A), K cos (A−B), in the equation to the bisector of A, viz., $\beta - \gamma = 0$, we get at once C−A = A−B, whence A = 60°.]

9222. (Sarah Marks, B.Sc.)—If a, b, c be positive quantities which are not all equal to each other, prove that

$$(b+c-a)(b-c)^2 + (c+a-b)(c-a)^2 + (a+b-c)(a-b)^2$$

will be a positive quantity.

Solution by H. L. ORCHARD, B.Sc., M.A.; D. WATSON, M.A.; *and others.*

Let a be that quantity which is not greater than either of the others, and $b = a+x$, $c = a+y$. Then the given expression becomes, on substituting and arranging, $2\{(x-y)(bx-cy)+axy\}$; and, whatever be the relative values of x and y, this is easily seen to be a positive quantity.

9286. (H. L. ORCHARD, B.Sc., M.A.)—Jack and Jill go a journey of eighty miles; the one on a tricycle, the other on a bicycle. Jack starts first, and, at the end of four hours, is overtaken by Jill. Jill completes the course in $2\frac{1}{2}$ hours, and Jack completes it in the same time which Jill took to overtake him. Assuming Jack and Jill to travel uniformly, find their speeds.

Solution by W. R. SEYMOUR; R. KNOWLES, B.A.; *and others.*

Let x, y be the number of miles per hour at which Jack and Jill respectively travel; then $80-4x = 9y/4 = 4x^2/y$, and $x = 11\frac{3}{7}$, and $y = 15\frac{5}{11}$.

8706. (Rev. T. R. TERRY, M.A.)—Find the eccentricity of the conic

$$27x^2-24xy+20y^2-30x-88y+131 = 0,$$

and determine the position of its foci and directrices.

Solution by the PROPOSER; R. W. D. CHRISTIE, M.A.; *and others.*

For the eccentricity, we have $(e^2-\frac{25}{36})(e^2+\frac{25}{11}) = 0$, therefore $e = \frac{5}{6}$. Whence, comparing terms of second degree, the equation may be put in the form

$$(x-\alpha)^2+(y-\beta)^2 = (\tfrac{5}{6})^2\left(\frac{3x+4y-p}{5}\right)^2;$$

comparing other terms, we get

$$\alpha = 1 \text{ or } \tfrac{36}{11},\quad \beta = 2 \text{ or } \tfrac{64}{11},\quad p = 7 \text{ or } \tfrac{365}{11}.$$

Therefore foci are (1, 2) and $(\frac{36}{11}, \frac{64}{11})$; directrices are $3x+4y-7 = 0$, and $3x+4y-\frac{365}{11} = 0$.

9243. (Professor NILKANTHA SARKAR, M.A.)—Show that the asymptotes of an equilateral hyperbola are the loci of the extremities of lengths measured on the tangents, equal in length to the normals, starting from the point of contact.

Solution by R. F. Davis, M.A.; R. Knowles, B.A.; *and others.*

For, let the tangent at P be produced to R (for convenience sake suppose in direction AP) so that PR = the normal PG.

Then, since PG = CD, it follows that R lies on the asymptote. For the remote vertex of the parallelogram constructed on CP, CD as adjacent sides is known to lie on the asymptote.

8894. (J. M. Dyer, M.A.)—If p is a prime >3, show that

$$(p-1)!\ [1+\tfrac{1}{2}+\ldots\ldots 1/(p-1)]$$

is divisible by p^2. [Or, thus:—$d/dp\,(p-1!) = \mathrm{M}(p^2)$.]

Solution by R. W. D. Christie, M.A.; Sarah Marks, B.Sc.; *and others.*

Primes >3 are of the form $6\mathrm{N}\pm1$, and therefore odd.

We have $(p-1)!\,[1+\tfrac{1}{2}\ldots\ldots 1/p-1]$, and, reducing to a common denominator, we have

$$(p-1)!\left[\frac{(p-1)!+\tfrac{1}{2}(p-1)!\ldots\ldots(p-1)!\,/\,p-1}{(p-1)!}\right]$$

$$=[(p-1)!+\tfrac{1}{2}(p-1)!\ldots\ldots(p-1)!\,/\,p-1].$$

Now, by adding the first and last, the second and last but one, and so on, of these terms, we get

$$p\left[\frac{(p-1)!}{p-1}+\frac{(p-1)!}{2(p-2)}+\frac{(p-1)!}{3(p-3)}\ldots\ldots\frac{(p-1)!}{\tfrac{1}{2}(p-1)\,\tfrac{1}{2}(p+1)}\right].$$

Reducing again to a common denominator, we have

$$p(p-1)!\,[2(p-2)\,3(p-3)\ldots+(p-1)(p-3)\,3\ldots+(p-1)\,2(p-2)\ldots]$$

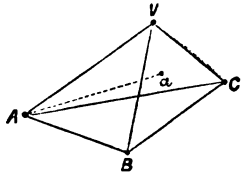
$$=p(p-1)!\,[p\mathrm{X}]=p^2[(p-1)!\,\mathrm{X}]=\mathrm{M}(p^2).$$

9251. (S. Tebay, B.A.)—In the tetrahedron VABC, the dihedral angles over the base ABC are X, Y, Z. If perpendiculars from A, B, C on the faces make angles X′, Y′, Z′ with VA, VB, VC, prove that

$$\sin\mathrm{X}\cos\mathrm{X}'=\sin\mathrm{Y}\cos\mathrm{Y}'=\sin\mathrm{Z}\cos\mathrm{Z}'.$$

Solution by R. F. Davis, M.A.

Let Aa be the perpendicular from A on the face VBC; ∇ = the volume of the tetrahedron.



Then $\cos\mathrm{X}'=\cos\mathrm{VA}a=\mathrm{A}a/\mathrm{VA}$.

But $3\nabla=\mathrm{C}c\,.\,\triangle\mathrm{VAB}$

$$=\tfrac{1}{2}\mathrm{C}c\,.\,\mathrm{VA}\,.\,\mathrm{B}b\,\mathrm{cosec}\,\mathrm{X},$$

therefore

$$\cos\mathrm{X}'\sin\mathrm{X}=\mathrm{VA}\,.\mathrm{VB}\,.\mathrm{VC}\,/\,6\nabla$$

$$=\text{a symmetrical quantity.}$$

[Let ϕ_1, ϕ_2, ϕ_3 be the perpendiculars from A, B, C on the opposite face A_1, A_2, A_3; then, since $\phi_1 = (2A_2 \sin Z)/c$, we have $\phi_1\phi_2 = (6V \sin Z)/c$; hence $\phi_1\phi_2\phi_3/6V = (\phi_3 \sin Z)/c = (\phi_2 \sin Y)/b = (\phi_1 \sin X)/a$
$= \sin X \cos X' = \sin Y \cos Y' = \sin Z \cos Z'$.]

9226. (J. White.)—Prove that
$1^3 + 2^3 + 3^3 \ldots M^3$ is a factor of $(1^5 + 2^5 + 3^5 \ldots M^5) \times 3$.

Solution by Prof. Plamenewsky; R. Knowles, B.A.; *and others.*

Since $2S_5 = 3S_2^2 - S_3$ and $S_3 = S_1^2$,

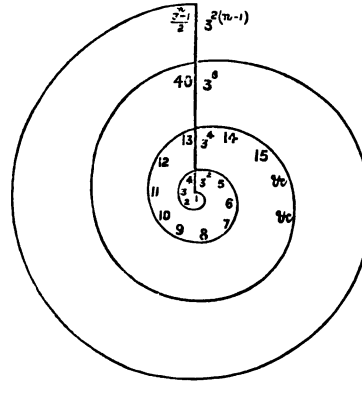
$$S_2 = S_1 \frac{2M+1}{3}, \quad S_2^2 = S_1^2 \frac{(2M+1)^2}{9} = \frac{S_3}{9}(2M+1)^2.$$

Hence $$3S_5 = S_3(2M^2 + 2M - 1).$$

9252. (R. W. D. Christie, M.A.)—Prove that the series
$[1 + 3^2 + 3^4 \ldots\ldots 3^{2(n-1)}]$
equals a sequence from unity $(1 + 2 + 3 \ldots\ldots n)$.

Solution by R. Knowles, B.A.; Prof. Beni Madhav Sarkar; *and others.*

$1 + 3^2 + 3^4 + \ldots\ldots 3^{2(n-1)} = \frac{1}{8}(3^{2n} - 1) = \frac{1}{2}\{\frac{1}{2}(3^n - 1)\}\{\frac{1}{2}(3^n + 1)\}$.



But $\frac{1}{2}(3^n - 1)$ and $\frac{1}{2}(3^n + 1)$ are consecutive integers; hence the series is of the form $\frac{1}{2}y(y+1)$. [The Proposer sends the above diagram in illustration of his problem.]

8652. (R. CURTIS, M.A.)—If $x_1y_1z_1$, $x_2y_2z_2$, $x_3y_3z_3$, $x_4y_4z_4$ are the coordinates of the four vertices of a tetrahedron (axes of coordinates being oblique), and ϕ = the solid angle contained by the three coordinate planes, show that

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \times \sin\phi = \text{volume of tetrahedron.}$$

Solution by the PROPOSER.

Suppose that $X = 0$, $Y = 0$, $Z = 0$ are three coordinate planes at mutual right angles, and that $\chi = 0$, $\psi = 0$, $\zeta = 0$, $\omega = 0$ are four planes of which the equations are $X\cos\alpha' + Y\cos\beta' + Z\cos\gamma' - p' = 0$,

$X\cos\alpha'' + \ldots - p'' = 0$, $X\cos\alpha''' + \ldots - p''' = 0$, $X\cos\alpha^{iv} + \ldots - p^{iv} = 0$.

Therefore

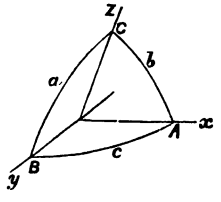
$$\begin{vmatrix} \chi_1 & \psi_1 & \zeta_1 & \omega_1 \\ \chi_2 & \psi_2 & \zeta_2 & \omega_2 \\ \chi_3 & \psi_3 & \zeta_3 & \omega_3 \\ \chi_4 & \psi_4 & \zeta_4 & \omega_4 \end{vmatrix} \equiv \begin{vmatrix} X_1 & Y_1 & Z_1 & 1 \\ X_2 & Y_2 & Z_2 & 1 \\ X_3 & Y_3 & Z_3 & 1 \\ X_4 & Y_4 & Z_4 & 1 \end{vmatrix} \begin{vmatrix} \cos\alpha' & \cos\beta' & \cos\gamma' & -p' \\ \cos\alpha'' & \cos\beta'' & \cos\gamma'' & -p'' \\ \cos\alpha''' & \cos\beta''' & \cos\gamma''' & -p''' \\ \cos\alpha^{iv} & \cos\beta^{iv} & \cos\gamma^{iv} & -p^{iv} \end{vmatrix}$$

But, placing ω at ∞, $\omega_1 = \omega_2 = \omega_3 = \omega_4 = -p^{iv} = \infty$, and therefore dividing the last columns of the first and third determinants, and

$$\begin{vmatrix} \chi_1 & \psi_1 & \zeta_1 & 1 \\ \chi_2 & \psi_2 & \zeta_2 & 1 \\ \chi_3 & \psi_3 & \zeta_3 & 1 \\ \chi_4 & \psi_4 & \zeta_4 & 1 \end{vmatrix} = 6 \text{ Vol. of Tetr.} \times (\cos\alpha', \cos\beta'', \cos\gamma''').$$

If a, b, c are the angles between the coordinate axes; A, B, C those between the coordinate planes,

$$(\cos\alpha', \cos\beta'', \cos\gamma''')^2 = \begin{vmatrix} 1 & -\cos C & -\cos B \\ -\cos C & 1 & -\cos A \\ -\cos B & -\cos A & 1 \end{vmatrix} = (\sin a \sin B \sin C)^2,$$



as χ_1 = perpendicular on plane $yz_1 = x_1 \sin B - \sin C$, &c.,

$$\therefore \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \sin A \sin B \sin C \,.\, \sin a \sin b \sin c = 6 \text{ Vol.} \times \sin a \sin B \sin C,$$

therefore $$\text{Vol.} = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ \ldots & \ldots & \ldots & \ldots \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \times \frac{\sin A \sin b \sin c}{6}.$$

But, if l', m', n', l'', m'', n'', l''', m''', n''' be the direction cosines of the three axes of coordinates with respect to a set of rectangular axes,

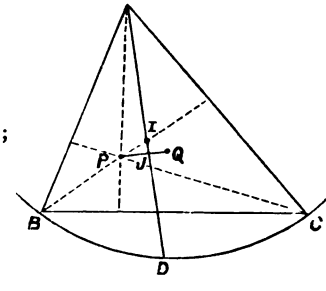
$$(l', m'', n'')^2 = \begin{vmatrix} 1 & \cos c & \cos b \\ \cos c & 1 & \cos a \\ \cos b & \cos a & 1 \end{vmatrix} = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c$$

$$= (\sin A \sin b \sin c)^2,$$

and (l', m'', n''') is $\sin\phi$ by definition.

Therefore Vol. of tetrahedron $= \frac{\sin\phi}{6} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$

8890. (Major H. W. L. Hime, R.A.)—If ABC be any triangle, I the in-centre, P the orthocentre and Q the circumcentre, and if AI meets PQ in J, prove (1) that $PJ/JQ = 2\cos A$; and hence (2) that if $A = 60°$ J is the mid-centre (*i.e.*, centre of the nine-point circle).



Solution by Professor Beyens, M.A.; *the* Proposer; *and others.*

1. $\frac{PJ}{JQ} = \frac{AP}{QD} = \frac{b\cos A \operatorname{cosec} B}{\frac{1}{2}b \operatorname{cosec} B}$

$= 2\cos A.$

2. If $A = 60°$, $2\cos A = 1$,

and $PJ = JQ$.

9237. (Professor Nilkantha Sarkar, M.A.) — Find three plane right-angled triangles whose areas are equal.

Solution by Artemas Martin, LL.D.

Let p^2-q^2, $2pq$, p^2+q^2; p^2-r^2, $2pr$, p^2+r^2; s^2-p^2, $2ps$, p^2+s^2, denote the sides of the triangles respectively. Then we must find

$$pq(p^2-q^2) = pr(p^2-r^2), \text{ and } pq(p^2-q^2) = ps(s^2-p^2);$$

or $$q(p^2-q^2) = r(p^2-r^2) \text{ and } q(p^2-q^2) = s(s^2-p^2) \ldots\ldots\ldots (1, 2).$$

From (1) and (2), we find

$$p^2 = r^2 + rq + q^2 = s^2 - sq + q^2, \text{ therefore } q = s - r \ldots\ldots\ldots (3).$$

Substituting in (3), $p^2 = s^2 - rs + r^2$ ..... (4).

(4) is satisfied by $r = m^2 - n^2$, $s = 2mn + m^2$,

$p = m^2 + mn + n^2$, therefore $q = 2mn + n^2$,

and the sides of the triangles are

$(m^2 - n^2)(m^2 + 2mn)$, $2(m^2 + mn + n^2)(2mn + n^2)$, $(m^2 + mn + n^2)^2 + (2mn + n^2)^2$;

$(2m^2 + mn)(2n^2 + mn)$, $2(m^2 - n^2)(m^2 + mn + n^2)$, $(m^2 - n^2)^2 + (m^2 + mn + n^2)^2$;

$(mn - n^2)(2m^2 + 3mn + n^2)$, $2(m^2 + 2mn)(m^2 + mn + n^2)$,

$(m^2 + mn + n^2)^2 + (m^2 + 2mn)^2$.

Take $m = 2$, $n = 1$, and the required triangles are

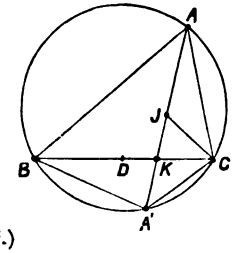
24, 70, 74; 40, 42, 58; 15, 112, 113.

Take $m = 3$, $n = 1$, and the triangles are

120, 182, 218; 105, 208, 233; 56, 390, 394; &c.

9276 (E. Vigarié.)—Dans tout quadrilatère harmonique, (1) la somme des inverses des carrés des côtés est égale à deux fois l'inverse de la puissance du point de Lemoine par rapport au cercle circonscrit; (2) le produit des côtés est égal à quatre fois le produit des droites qui joignent un sommet au milieu de la diagonale qui ne passe pas par ce sommet.

Solution by R. F. Davis, M.A.



Let ABC be a triangle inscribed in a circle; AA′ the symmedian line through A meeting BC in K and the circumference again in A′. Join A′B, A′C. Then ABA′C is a harmonic quadrilateral whose Lemoine point K is the intersection of the diagonals AA′, BC. Let D, J be the middle points of BC, AA′.

Since A′B . AC = A′C . AB, each = ½AA′. BC, or A′B = (a/b) JA, A′C = (a/c) JA. Since AA′, AD are inverse lines with respect to the angle A, the triangles ABD, AA′C are similar; hence AA′. AD = AB . AC, or $JA^2 = b^2c^2/(2b^2 + 2c^2 - a^2)$.)

Now the equation to AA′ being $c\beta = b\gamma$, therefore BK : CK = $c^2 : b^2$, and BK = $ac^2/(b^2 + c^2)$ and CK = $ab^2/(b^2 + c^2)$.

Thus BK . CK = "puissance de K par rapport au cercle circonscrit"
= $a^2b^2c^2/(b^2 + c^2)^2$.

The sum of the reciprocals of the squares of the sides

= $(b^2 + c^2)/b^2c^2 + (b^2 + c^2)/a^2 . JA^2 = 2(b^2 + c^2)^2/a^2b^2c^2$.

Product of the sides = $bc . (a/b)JA . (a/c)JA = a^2JA^2$. Since the triangles ABJ, CAJ are similar (J being the permanent centre of similitude of similar figures described upon CA, AB), therefore JB . JC = JA^2.

Also, D being the middle point of the symmedian line BC in the triangle ABA′, we have similarly DA . DA′ = $\frac{1}{4}a^2$. Hence

JB . JC . DA . DA′ = $\frac{1}{4}a^2 . JA^2$.

8965. (W. J. C. Sharp, M.A.)—Show that, if

$$x^n - P_1 x^{n-1} + P_2 x^{n-2} - P_3 x^{n-3} + \&c. = 0$$

be the equation whose roots are the squares of those of

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - p_3 x^{n-3} + \&c. = 0,$$

$$P_r = \frac{1}{2(n-r)!}\left\{\frac{p_{n-1}}{p_n}\left(\frac{d}{dp_1} + p_1\frac{d}{dp_2} + p_2\frac{d}{dp_3} + \&c.\right)\right.$$
$$\left. - \left(n\frac{d}{dp_2} + (n-1)\,p_1\frac{d}{dp_3} + (n-2)\,p_2\frac{d}{dp_4} + \&c.\right)\right\}^{n-r}(p_n^2).$$

Solution by Professor Sebastian Sircom, M.A.

Let $a_1, a_2 \ldots a_n$ be the roots of $x^n - p_1 x^{n-1} + p_2 x^{n-2} - p_3 x^{n-3} + \ldots = 0$,

then $\Sigma\left(\frac{dp_n}{da}\right) = \left(\frac{d}{da_1} + \frac{d}{da_2} + \ldots + \frac{d}{da_n}\right)p_n = p_{n-1}$, $\Sigma\left(\frac{dp_{n-1}}{da}\right) = 2p_{n-2}$,

and so on, then $$\Sigma\left(\frac{dp_{n-r+1}}{da}\right) = rp_{n-r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

whence $p_r = \frac{1}{(n-r)!}\left(\frac{d}{da_1} + \frac{d}{da_2} + \ldots\right)^{n-r} p_n$. Also $P_n = p_n^2$, then

$$P_r = \frac{1}{(n-r)!}\left(\frac{d}{d(a_1^2)} + \ldots\right)^{n-r}(p_n^2)$$
$$= \frac{1}{2(n-r)!}\left(\frac{1}{a_1}\frac{d}{da_1} + \frac{1}{a_2}\frac{d}{da_2} + \ldots\right)^{n-r}(p_n^2) \ldots\ldots\ldots\ldots (2).$$

To transform this, we have

$$\frac{d}{da_1} = \frac{dp_1}{da_1}\frac{d}{dp_1} + \frac{dp_2}{da_1}\frac{d}{dp_2} + \ldots, \text{ so for } a_2, a_3, \&c.,$$

then $$\Sigma\left(\frac{1}{a}\frac{d}{da}\right) = \Sigma\left(\frac{1}{a}\frac{dp_1}{da}\right)\frac{d}{dp_1} + \Sigma\left(\frac{1}{a}\frac{dp_2}{da}\right)\frac{d}{dp_2} + \ldots \quad \ldots\ldots (3);$$

but $$\frac{dp_r}{da_1} = p_{r-1} - a_1\frac{dp_{r-1}}{da_1},$$

whence

$$\Sigma\left(\frac{1}{a}\frac{dp_r}{da}\right) = p_{r-1}\Sigma\left(\frac{1}{a}\right) - \Sigma\left(\frac{dp_{r-1}}{da}\right) = \frac{p_{n-1}}{p_n}p_{r-1} - (n-r+2)\,p_{r-2}$$

from (1). Substituting in (3) and then in (2), we obtain the required result. The process can evidently be easily extended to cubes and higher powers.

8546. (R. Knowles, B.A.)—Tangents from a point T (hk) meet the ellipse $a^2y^2 + b^2x^2 = a^2b^2$ in P and Q, a third tangent at R meets these in M, N respectively; show that, if θ be the eccentric angle of the point R, the area of the triangle TMN is

$$(a^2k^2 + b^2h^2 - a^2b^2)^{\frac{1}{2}}\,(ak\sin\theta + bh\cos\theta - ab)\,/\,(ak\sin\theta + bh\cos\theta + ab).$$

Solution by G. G. Storr, B.A.; Rev. T. Galliers, M.A.; *and others.*

The equations to the tangents at P and R are

$$bx\cos\phi_1 + ay\sin\phi_1 - ab = 0, \quad bx\cos\theta + ay\sin\theta - ab = 0,$$

therefore coordinates of M are

$$\frac{x}{a\cos\frac{1}{2}(\phi_1+\theta)} = \frac{y}{b\sin\frac{1}{2}(\phi_1+\theta)} = \frac{1}{\cos\frac{1}{2}(\phi_1-\theta)},$$

and
$$MN = \frac{\sin\frac{1}{2}(\phi_2-\phi_1)}{\cos\frac{1}{2}(\phi_1-\theta)\cos\frac{1}{2}(\phi_2-\theta)}\left\{a^2\sin^2\theta + b^2\cos^2\theta\right\}^{\frac{1}{2}},$$

therefore
$$\Delta TMN = \tfrac{1}{2}\frac{bh\cos\theta + ak\sin\theta - ab}{(b^2\cos^2\theta + a^2\sin^2\theta)^{\frac{1}{2}}} \cdot \frac{\sin\frac{1}{2}(\phi_2-\phi_1)}{\cos\frac{1}{2}(\phi_1-\theta)\cos\frac{1}{2}(\phi_2-\theta)} \times \left\{a^2\sin^2\theta + b^2\cos^2\theta\right\}^{\frac{1}{2}},$$

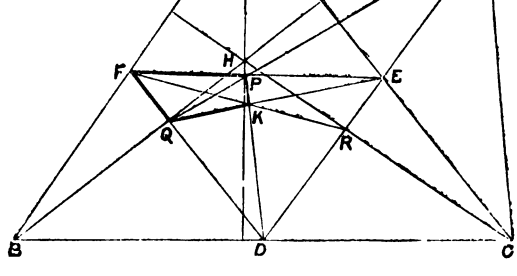
and remembering that $\frac{h^2}{a^2} + \frac{k^2}{b^2} = \left(\frac{1}{\cos^2\frac{1}{2}(\phi_1-\phi_2)}\right)$

$$= \tfrac{1}{2}\frac{(bh\cos\theta + ak\sin\theta - ab)(b^2h^2 + a^2k^2 - a^2b^2)^{\frac{1}{2}}}{(b^2h^2+a^2k^2)^{\frac{1}{2}}\cos\frac{1}{2}(\phi_1-\theta)\cos\frac{1}{2}(\phi_2-\theta)} = \text{result.}$$

9181. (Sarah Marks, B.Sc.)—The mid-points of the perpendiculars drawn from A, B, C on the opposite sides of the triangle ABC are P, Q, R; also D, E, F are the mid-points of BC, CA, AB; prove that (1) PQ, DE and the fourth harmonic to CB, CR, CA meet in a point; and (2) the three such points lie on a straight line.

Solution by Professor Schoute; R. F. Davis, M.A.; *and others.*

It is known that the joins DP, EQ, FR meet in the Lemoine-point K. Of the quadrangle FPKQ the pairs of opposite sides pass through D and



E, and the diagonals through R and R′; *i.e.*, D, R, E, R′ are four harmonic points. Therefore the join CR′ of C and the point R′ common to DE and PQ is fourth harmonic to CB, CR, CA.

As in triangle DEF, the lines DP, EQ, FR are concurrent in K, and EF, FD, DE are divided harmonically by PP′, QQ′, RR′ respectively; the points P′, Q′, R′ lie on the polar of K with respect to the triangle DEF.

9147. (S. Constable.)—The vertex of a triangle is fixed, and the vertical angle given, while the base angles move on two fixed lines. Perpendiculars are let fall from the vertex on the two fixed lines, and the feet of these perpendiculars are joined to the mid-points of the sides of the triangle; prove that the locus of the point where these lines intersect is a circle.

Solution by D. O. S. Davies, M.A.; Belle Easton; *and others.*

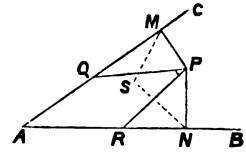
If AB, AC be the fixed straight lines, and P the vertex of the variable triangle PQR, we have

$$\angle MSN + SMP + MPN + PNS = 2\pi,$$

$$\angle MSN + 2MPN - QPR = 2\pi,$$

therefore $\angle MSN$ = constant;

hence the locus of S is a circle passing through M, N.



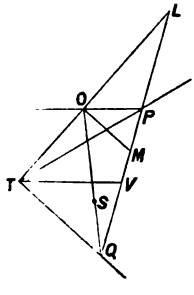
9131. (W. J. Barton, M.A.)—If tangents at P and Q to a parabola meet at T, SQ meets the diameter through P and O, PQ meets OT in L, and OM perpendicular to OT in M; show that LV. MQ = PV. LQ, V being middle point of PQ.

Solution by D. O. S. Davies, M.A.; R. Knowles, B.A.; *and others.*

The diameter through T bisects PQ, and therefore also bisects OQ. TQ is equally inclined to TV and OQ. Hence $O\widehat{T}Q$ is a right angle.

Therefore $$\frac{LQ}{MQ} = \frac{LT}{OT} = \frac{LV}{PV}.$$

Hence $$LV.MQ = PV.LQ.$$



9178. (D. O. S. Davies, M.A.)—If P is any point on a rectangular hyperbola whose centre is C, prove that the envelope of the circles whose diameters are CP, is the inverse of the hyperbola with regard to its auxiliary circle.

Solution by Profs. Beni Madhav Sarkar, B.A.; Beyens; *and others.*

Let the coordinates of P be $a\sec\phi$, $a\tan\phi$. The equation of the circle on CP as diameter, is

$$x(x-a\sec\phi)+y(y-a\tan\phi)=0 \quad \ldots\ldots\ldots\ldots\ldots (1).$$

Differentiating with respect to ϕ,

$$-ax\tan\phi\sec\phi-ay\sec^2\phi=0,$$

that is, $$x\sin\phi+y=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Eliminating ϕ between 1 and 2,

$$x^2+y^2=\frac{ax^2}{(x^2-y^2)^{\frac{1}{2}}}=\frac{ay^2}{(x^2-y^2)^{\frac{1}{2}}},$$

or $(x^2+y^2)^2=a^2(x^2-y^2)$, which is the *Lemniscata* of Bernoulli, and the inverse of the rectangular hyperbola with respect to the auxiliary circle.

9273. (Professor Bordage.)—Solve the equation

$$7346\times 7^{\sec x}+7^{1+\sec x}-7010\times 7^{2\sec x}-7^{3+2\sec x}+3\times 7^{2+3\sec x}=147.$$

Solution by W. J. Greenstreet, B.A.; Paul Kleeberger; *and others.*

Putting $7^{\sec x}=y$ throughout, we get

$$147(y^3-1)-7353y(y-1)=0,$$

which gives $y=1$, $\frac{1}{49}$ or 49; hence $\sec x=0$, -2, $+2$; therefore

$$x=2n\pi\pm\tfrac{1}{3}\pi.$$

9039. (Professor Chakravarti, M.A.)—If

$$ab(ab^{-1}x+a^{-1}by)=(x+y)(z^2+xy),$$ prove that (1)

$$x(a+y-z)(a+y+z)+y(b+x-z)(b+x+z)=2xy(a+b+x+y);$$

and (2)
$$\frac{b+z-x}{b+z+x}\cdot\frac{a+z-y}{a+z+y}=\frac{a+b-x-y}{a+b+x+y}.$$

Solution by W. J. Greenstreet, B.A.; Sarah Marks, B.Sc.; *and others.*

From (1) we have $z^2=\{a^2x+b^2y-(x+y)xy\}/(x+y)$,

and
$$x(a+y-z)(a+y+z)+y(b+x-z)(b+x+z)$$
$$=a^2x+b^2y+xy(x+y)+2xy(a+b)-(x+y)z^2=2xy(a+b+x+y).$$

Again, $(b+z-x)(a+z-y) = z^2+(b-x)(a-y)+z(a+b-x-y)$,
which becomes $= (a+b-x-y)(ax+by)/(x+y)+z(a+b-x-y)$
$= (a+b-x-y)\{(ax+by)/(x+y)+z\}$.

Similarly $(b+z+x)(a+z+y) = (a+b+x+y)\{(ax+by)/(x+y)+z\}$;
therefore

$$(b+z-x)(a+z-y)/(b+z+x)(a+z+y) = (a+b-x-y)/(a+b+x+y).$$

9237. (J. White.)—If D be the circumcentre and E the incentre of the triangle ABC, prove that (1) if DE passes through the orthocentre,

$$\cos^2 A \cos B + \cos^2 B \cos C + \cos^2 C \cos A = \cos^2 A \cos C + \cos^2 B \cos A + \cos^2 C \cos B,$$

and (2) if this condition is satisfied, the incentre, the circumcentre, and the centre of the nine-points circle of the triangle are collinear.

Solution by W. J. Greenstreet, B.A.; R. Knowles, B.A.; *and others.*

The equation to the join of the incentre and circumcentre is

$$\alpha(\cos B - \cos C) + \beta(\cos C - \cos A) + \gamma(\cos A - \cos C) = 0.$$

This equation must be satisfied by the coordinates of the orthocentre, which are as sec A : sec B : sec C; therefore

$$\frac{\cos B - \cos C}{\cos A} + \ldots = 0, \text{ or } \cos^2 B \cos C - \cos^2 C \cos B + \ldots = 0,$$

the required expression. (2) The circumcentre, orthocentre, and nine-points centre are always collinear; therefore, when above condition is satisfied, the incentre, circumcentre, orthocentre, and nine-points centre are collinear.

9187. (J. White.)—Prove that $8 \cos \frac{3}{7}\pi \cos \frac{2}{7}\pi \cos \frac{1}{7}\pi = 1$.

Solution by Profs. Beni Madhav Sarkar, B.A.; Beyens; *and others.*

$$8 \cos \tfrac{3}{7}\pi \cos \tfrac{2}{7}\pi \cos \tfrac{1}{7}\pi = \frac{8 \cos \frac{3}{7}\pi \cos \frac{2}{7}\pi \cos \frac{1}{7}\pi \sin \frac{1}{7}\pi}{\sin \frac{1}{7}\pi}$$

$$= \frac{2 \cos \frac{3}{7}\pi \sin \frac{4}{7}\pi}{\sin \frac{1}{7}\pi} = \frac{\sin \frac{1}{7}\pi + \sin \pi}{\sin \frac{1}{7}\pi} = 1.$$

APPENDIX I.

A BRIEF EXPLANATION OF THE ADVANTAGES TO BE DERIVED FROM USING THE *AID TO APPROXIMATE CALCULATION.*

By D. Biddle.

There are three chief characteristics of these logarithmic tables:—

(1) They are presented in one view, and therefore require no turning of pages;

(2) The conversion of numbers into logarithms and the resolution of logarithms into numbers are effected by an identical process, only on different parts of the sheet;

(3) The required figures in any search are found almost instantaneously, by the division into sets, columns, and rows.

Owing to having the whole before our eyes at once, it is quite possible to perform many minor logarithmic operations mentally, that is, without having recourse to pen or pencil. A pointer of some sort (even a finger) being placed at each of two *mantissæ*—logarithmic decimals—where found, it is easy to supply the characteristics, and add or subtract as required, without transcription. But, unless the memory is very good, this plan is not to be recommended.

Where the numbers do not exceed three figures each, considerable accuracy may be attained in the result on all occasions, even though several numbers enter into the calculation. Thus, multiply 62·7 by ·384, and this again by 2·41. Here,

$$1{\cdot}797268 + \bar{1}{\cdot}584331 + 0{\cdot}382017 = 1{\cdot}763616 = \log 58{\cdot}025.$$

The following, again, is an instance of a kind in which great accuracy is attainable by the use of these tables, and in which the process is reduced in length by taking the complementary figures of logarithms which would otherwise be subtracted, in order to make the operation one of addition only. But in all such cases, care must be taken to deduct one from the next higher column after reaching the beginning of each co-logarithm, whilst adding.

Find the value of £1 in *francs*, supposing it known that 3150·58 francs = 1000 grammes, that 31·1 grammes = 1 oz. troy, that 1 oz. troy of standard gold = 77·875 shillings, and that 20 shillings = £1.

By Table I.

log 3·15	= 0·498311
05	69
8	11
log 31·1	= 1·492760
log 20·	= 1·301030
co-log 77·8	= 8·109020
7	609
5	72
	1·400782

By Table II.

1·400	= log 25·1189
7	410
8	47
2	1
	25·1647

By taking here parts of the *actual* instead of the *average* difference, the result is 25·16415, which is nearly exact.

A further advantage to be derived from using the *Aid to Approximate Calculation* will be seen whilst considering the problem enunciated as follows:—

(A.) *How to find the logarithm of the sum or difference of two numbers of which we have the respective logarithms, without resolving these logarithms into those numbers.*

It is very inconvenient to be diverted from a long and composite logarithmic calculation, by having midway to find the numbers corresponding to certain logarithms, when the logarithm of the sum or difference of the numbers is intermediately required. Thus, to enumerate what ordinarily takes place in such cases: the number corresponding to one of the logarithms is searched for; next, the number corresponding to the other logarithm; and, the two numbers being separately recorded in the usual relative positions, the one is added to, or subtracted from, the other; but this is not all, for the logarithm of the resulting sum or difference must be searched for and recorded, before the interrupted calculation can be resumed.

This interruption may be greatly lessened by attention to the following considerations:—

$$10^a + 10^b = 10^b (10^{a-b} + 1), \quad 10^a - 10^b = 10^b (10^{a-b} - 1).$$

In other words, it is only necessary (1) to obtain the difference of the two given logs.; (2) to find the number corresponding to this difference, and again (3), the log. of the same number increased or diminished by unity. (4) This log., added to the log. represented by b, gives the log. of the sum or difference required. Not only is the labour lessened, but there are fewer chances of error. Instead of three searches, three records, and an addition or subtraction needing a fourth record, we have an initial subtraction and record, a first search needing no record, the addition of unity (done mentally), a second search and record, and an addition needing a third record, which however contains the figures required.

The process is extremely simple, if the tables in the *Aid to Approximate Calculation* be used. For, having obtained the difference of the two given logs., and marked the position of the six figures in Table I. which are nearest to it (lower in value by preference), we can find the log. of the corresponding number increased or diminished by unity, by simply transferring the marker to the same place in the next (for the sum) or preceding (for the difference) set, or column, or row, according as the characteristic of the logarithmic difference is 0·, or 1·, or 2·. In the case of higher characteristics, recourse must be had to the *parts* at the foot of the same column. Where a negative characteristic occurs in such logarithmic differences, it is best to find the corresponding number, and, after adding unity, search again.

Examples.

1. Find the log. of the sum of two numbers whose logs. are respectively 3·285557 (= log 1930) and 2·123852 (= log 133).

The difference of these logs. is 1·161705, which is found to exceed the log. of 14·5 by 337, that is, to be the log. of 14·5112. In the next column we find the log. of 15·5 to be 1·190332

1	280
1	28
2	6
	1·190646, which added to the lesser of
the two given logs.	2·123852
gives	3·314498 = log 2063.

2. Find the log. of the difference of two numbers whose logs. are respectively 4·895069 (= log 78536) and 2·661813 (= log 459).

In this case, the difference of the logs. is 2·233256, which is found to exceed the log. of 171 by 260. The number diminished by unity having its log. in the preceding row, the same excess,

	260
may be added to the log. of 170,	2·230449
and the lesser given log.	2·661813
yielding	4·892522 = log 78077.

Without attempting to exhaust the uses to which the "Aid" may be put, particular mention should be made of its value in the calculation of *percentages*, where the answer need not exceed four figures. Thus, What percentage is 159 of 2174?

By Table I.		By Table II.	
log 15900·	= 4·201397	0·864	= log 7·31139
co-log 217	= 6·663540	1	169
4	192	3	50
giving, by addition,	0·864129	say,	7·314

After considerable practice, it becomes possible without difficulty to adjust the parts given at the foot of each column, so as to make them suit any particular row, with nearly as much accuracy as if they were taken from the actual difference. This is done by looking to the corresponding part on the left or on the right of the particular column, according as the row belonging to the third figure of the convertend is above or below the fifth; by mentally dividing into ten the difference between the two average parts thus compared; and by adding to, or subtracting from, the average part in the original column, as many of those tenths as are indicated by the difference between the third figure of the convertend and 5.

Thus, in converting 2734 into logarithms, we have

$$\begin{aligned} \log 273 &= 3{\cdot}436163 \\ \tfrac{4}{10}(656-632) &= \underline{\quad 5} \\ \text{giving} \quad & 3{\cdot}436800, \end{aligned} \qquad \begin{matrix} \\ 4 \\ \\ \end{matrix} \qquad \begin{matrix} \\ 632 \\ \\ \end{matrix}$$

which differs but little from 3·4367985 given in the seven-figure tables. But the actual difference gives 3·436798.

The "Aid" will be of great service in the earlier stages of problems such as the following, more voluminous tables of logarithms not being really needed until near the close of the process.

(B.) *How, in equations of high degree, to find a second root where one is given.*

The best way of explaining the method will be to give, as an example, the following solution of Question 9185 :—

9185. (Major-General P. O'Connell.) Mr. Thomas Weddle, by a solution extending over six printed pages, has found one root of $1379664x^{622} + 2686034 \times 10^{432} \times x^{153} - 1729024 \times 10^{575} \times x^{60} + 2524156 \times 10^{574} = 0$ to be $x = 8{\cdot}367975431$; it is required to find, by using a seven-figure table of common logarithms, another positive root containing seven figures, the solution occupying not more than one octavo page.

Rendering the equation thus,

$$ax^m + bx^n - cx^p + d = 0,$$

we have $\log a = 6{\cdot}1397733$, $\log b = 438{\cdot}4291115$, $\log c = 525{\cdot}2378006$,

$\log d' = 580{\cdot}4021162$, and $\log x_1 = 0{\cdot}9226204$.

It is also clear that

$$a(x_1^m - x_2^m) + b(x_1^n - x_2^n) - c(x_1^p - x_2^p) = 0.$$

Let $y = x_1/x_2$, then $\log x_2 = \log x_1 - \log y$.

Accordingly, by the process of transformation already explained, and putting K ≡ the number corresponding to the logarithm in the bracket immediately after, we have

$$\begin{aligned} &K\left[\log a + m\log x_1 - m\log y + \log\left\{K(m\log y)-1\right\}\right] \\ &+K\left[\log b + n\log x_1 - n\log y + \log\left\{K(n\log y)-1\right\}\right] \\ &-K\left[\log c + p\log x_1 - p\log y + \log\left\{K(p\log y)-1\right\}\right] = 0. \end{aligned}$$

Let $\log a + m\log x_1 = A$, $\log b + n\log x_1 = B$, $\log c + p\log x_1 = C$, also

$m\log y = M$, $n\log y = N$, $p\log y = P$, $\log\left\{K(m\log y)-1\right\} = M'$,

$\log\left\{K(n\log y)-1\right\} = N'$, $\log\left\{K(p\log y)-1\right\} = P'$.

Then $K\left[A-(M-M')\right]+K\left[B-(N-N')\right]-K\left[C-(P-P')\right] = 0.$

Now $A' = 580{\cdot}0096621$, $B' = 579{\cdot}5900327$, $C' = 580{\cdot}5950246$; but we may curtail the characteristics by 579 each without affecting our equation, so let $A = 1{\cdot}0096621$, $B = 0{\cdot}5900327$, $C = 1{\cdot}5950246$. With these data, we proceed by a series of trials, taking r_1, r_2, r_3, &c. as representative of y, in succession, as shown in the following Synopsis A; and, so soon as we arrive at an error of a different sign from the last, we can readily hit upon a series, constantly approaching y, by taking the new log r between the two former, at distances proportioned to the two errors. Six trials reduce the amount of error to 0·000003, in which case $r_6 = y$ very closely. Therefore

$$\log x_2 = \log x_1 - \log r_6 = 0{\cdot}9226204 - 0{\cdot}00274984 = 0{\cdot}91987056,$$

and $x_2 = 8{\cdot}315159$.

(C.) *How, in equations of high degree, to find the roots independently.*

Here the same equation will serve, and may be rendered

$$ax^m + bx^n = cx^p - d.$$

This, by transformation, becomes

$$\begin{aligned} &\log\left\{K(\log a + m\log x - \log b - n\log x)+1\right\} + \log b + n\log x \\ &\quad = \log\left\{K(\log c + p\log x - \log d)-1\right\} + \log d; \end{aligned}$$

that is,

$$\begin{aligned} &\log\left\{K\left[(m-n)\log x - 432{\cdot}2893382\right]+1\right\} + 438{\cdot}4291115 + n\log x \\ &\quad = \log\left\{K(p\log x - 55{\cdot}1643156)-1\right\} + 580{\cdot}4021162. \end{aligned}$$

Each of the quantities within brackets gives us an approximate value of log x, with either of which we may start:—

$(m-n)\log x = 432{\cdot}2893382$, whence $\log x = 0{\cdot}921726$ .....(1),

$p\log x = 55{\cdot}1643156$, whence $\log x = 0{\cdot}919405$ .....(2).

The former is nearest the value of log x agreeing with Mr. Weddle's root. We will therefore take (2), and rectify according to the several errors, except after the first two trials which are of use only for the signs (see Synopsis B). The seventh and last trial shows that $\log x = 0{\cdot}91987057$ nearly, whence $x = 8{\cdot}315159$, as found by the previous method.

SYNOPSIS A.

$\log r$.	$m \log r$.	K ($m \log r$).	log [K ($m \log r$) − 1].	(log A − diff.)	K (log A − diff.).	Error.
1·000000	622·000	∞	622·000	1·0096621	10·22497	−25·241526
0·010000	6·220	1659586·5	6·2199997	1·0096618	10·22495	−15·470272
0·001000	0·622	4·187936	0·5035096	0·8911717	7·78343	+ 3·860195
0·0027972	1·739858	54·936127	1·7318797	1·0016838	10·03885	− 0·138028
0·00273499	1·701165	50·253346	1·6924357	1·0009328	10·02150	+ 0·043279
0·00274984	1·7104004	51·353447	1·7018567	1·0011184	10·025785	+ 0·000003

$\log r$.	$n \log r$.	K ($n \log r$).	log [K ($n \log r$) − 1].	(log B − diff.)	K (log B − diff.).
1·000000	153·000	∞	153·000	0·5900327	3·890744
0·010000	1·530	33·884408	1·5169910	0·5770237	3·775928
0·001000	0·153	1·422329	1·6256509	0·0626436	1·155270
0·0027972	0·427972	2·679004	0·2250517	0·3671124	2·438442
0·00273499	0·4184535	2·620918	0·2097610	0·3813402	2·406247
0·00274984	0·4207254	2·634665	0·2134288	0·3827361	2·413993

$\log r$.	$p \log r$.	K ($p \log r$).	log [K ($p \log r$) − 1].	(log C − diff.)	K (log C − diff.).
1·000000	60·000	∞	60·000	1·5950246	39·35724
0·010000	0·600	3·981072	0·4743725	1·4693971	29·47115
0·001000	0·060	1·148153	$\bar{1}$·1707105	0·7057351	5·078495
0·0027972	0·167832	1·471743	$\bar{1}$·6737055	1·1006981	12·615320
0·00273499	0·164099	1·459147	$\bar{1}$·6619518	1·0928774	12·384468
0·00274984	0·1649904	1·462145	$\bar{1}$·6647783	1·0948125	12·439775

SYNOPSIS B.

Trial $\log x$.	469 $\log x$.	K (469 $\log x$ − 432·2893382).	log [K (469 $\log x$ − 432·289) + 1].	(438·4291115 + 153 $\log x$).
0·919405	431·200945	0·081584	0·0340601	579·098077
0·920000	431·480000	0·155118	0·0626264	579·189112
0·919800	431·386200	0·124985	0·0511469	579·158512
0·919883	431·425127	0·136706	0·0556482	579·171211
0·9198718	431·4198742	0·135063	0·0550201	579·1694969
0·9198706	431·4193114	0·134888	0·0549532	579·1693133
0·91987057	431·4192973	0·134884	0·0549516	579·1693087

Trial $\log x$.	60 $\log x$.	K (60 $\log x$ − 55·1643156).	log [K (60 $\log x$ − 55·164) − 1].	Error after adding 580·4021162.
0·919405	55·1643156	1·000000	Infin. Neg.	−579·1321871
0·920000	55·2000000	1·085636	$\bar{2}$·9326564	+ 0·0830342
0·919800	55·1880000	1·056050	$\bar{2}$·7485756	− 0·0589671
0·919883	55·1929800	1·068229	$\bar{2}$·8339690	+ 0·0092260
0·9198718	55·1923080	1·066577	$\bar{2}$·8233242	+ 0·0009230
0·9198706	55·1922360	1·0664007	$\bar{2}$·8221727	+ 0·0000224
0·91987057	55·1922342	1·0663966	$\bar{2}$·8221459	+ 0·0000018

[The value of Mr. BIDDLE's method will be seen by comparing it with the PROPOSER's Solution of his Question 9185, as set forth hereunder :—

$$f(x) = 1379664x^{622} + 2686034 \times 10^{432} \times x^{153} - 17290224 \times 10^{518} \times x^{60} + 2524156 \times 10^{574} = 0;$$

or, dividing by 10^{580},

$$f(x) = 1379664 \times 10^{42}\left(\frac{x}{10}\right)^{622} + 2686034 \times 10^{5} \times \left(\frac{x}{10}\right)^{153} - 172902{\cdot}24\left(\frac{x}{10}\right)^{60} + 2{\cdot}524156 = 0.$$

If $y = \left(\frac{x}{10}\right)^{60}$, this equals

$$f(y) = 1379664 \times 10^{42} \times y^{10\cdot36} + 2686034 \times 10^{5} \times y^{2\cdot55} - 172902{\cdot}24y + 2{\cdot}524156 = 0.$$

As a first approximation, let $y = 1{\cdot}5 \times 10^{-5}$, this being nearly the result of rejecting the first two terms of the equation as being small.

Remember that y is supposed to be expressed as a factorial equal to $a_1(1+a_2)(1+a_3)\ldots$, and that the first set of figures is devoted to the computation of the logs. of the required powers of successive factors a_1, $1+a_2$, &c.

$$\log a_1 = \log 1{\cdot}5 - \log 10^5$$

$\log 1{\cdot}5 = \cdot1760913$

$10{\cdot}36\dot{} = 10 + \frac{1}{3} + \frac{1}{30}$

$10 \log 1{\cdot}5$	=	$1{\cdot}760913$
$\frac{1}{3}$,,	=	$\cdot0586971$
$\frac{1}{30}$,,	=	$\cdot00586971$
$\therefore\ 10{\cdot}36\dot{} \log 1{\cdot}5$	=	$1{\cdot}82547981$
$-10{\cdot}36\dot{} \log 10^5$	=	$\overline{51}{\cdot}83333333$
$\therefore\ \log a_1^{10\cdot36\dot{}}$	=	$\overline{51}{\cdot}99214648$

$2{\cdot}5 \log 1{\cdot}5$	=	$\cdot44022825$
$\cdot05$,,	=	$\cdot008804565$
		$\cdot449032815$
$2{\cdot}55 \log 10^5$	=	$12{\cdot}75$
$\therefore\ \log a_1^{2\cdot55}$		$\overline{13}{\cdot}699032815$

(1) ..... $\log(1+a_2) = \log 1{\cdot}0372$

	=	$\cdot0158625$
$2{\cdot}55 \log(1+a_2)$	=	$\cdot040449375$
$10{\cdot}36\dot{} \log(1+a_2)$	=	$\cdot16444125$

(2) .....$\log(1+a_3) = \log 1{\cdot}00064574$

	=	$\cdot0002804$
$2{\cdot}55 \log(1+a_3)$	=	$\cdot0007150$
$10{\cdot}36\dot{} \log(1+a_3)$	=	$\cdot0029067$

$\log a_1$	=	$\overline{5}{\cdot}1760913$
$\log(1+a_2)$	=	$\cdot0158625$
$\log(1+a_3)$	=	$\cdot0002804$
$\log y$	=	$\overline{5}{\cdot}1922342$
$\log \frac{1}{10}x = \frac{1}{60}\log y$	=	$\overline{1}{\cdot}91987057$
	=	$\log \cdot8315159$
$\therefore\ x$	=	$8{\cdot}315159$

	log 1379664 × 10^{42}	log 2686034 × 10^{5}	log 172902·24
	= 48·1397734	= 11·4291116	= 5·2378006
	51·9921465	13·6990328	5·1760913
(1) ...	$\bar{2}$·1319199	$\bar{1}$·1281444	·4138919
	·16444125	·0404494	·0158625
(2) ...	$\bar{2}$·29636115	$\bar{1}$·1685938	·4297544
	·00290670	·0007150	·0002804
(3) ...	$\bar{2}$·29926785	$\bar{1}$·1693088	·4300348

Below are the natural numbers corresponding to the logarithms above—

(1) ...	·01354940	·1343212	2·593533
(2) ...	·01978614	·1474327	2·690013
(3) ...	·01991901	·1476756	2·691751

Below are the computations of the values of the second and third factors of y—

(1) $-2\cdot59353\times1 = -2\cdot593$
$+\ \cdot01355\times10\cdot36 = \cdot140$
$+\ \cdot13432\times2\cdot55 = \cdot342$
$-2\cdot44566 \qquad -2\cdot111$
$+2\cdot52416$
$\cdot07850$

$$\frac{\cdot07850}{2\cdot111} = \cdot0372;$$

$\therefore$ 2nd factor of $y = 1\cdot0372$.

For continuation of the computation, see the part marked (1) on the foregoing page.

(2) $-2\cdot6900130\times1 = -2\cdot6900$
$+\ \cdot0197861\times10\cdot36 = \cdot2051$
$+\ \cdot1474327\times2\cdot55 = \cdot3760$
$-2\cdot5227942 \qquad -2\cdot1089$
$+2\cdot524156$

$$\frac{\cdot001368}{2\cdot1089} = \cdot00064574;$$

$\therefore$ 3rd factor of $y = 1\cdot00064574$.

(3) $-2\cdot691751$
$\cdot01991901$
$\cdot1476756$
$-2\cdot52415639$
$+2\cdot524156$
$-\ \cdot00000039$ = the final value obtained for $f(y)$, the true value being zero.]

The following Question furnishes also a good instance of the way in which the "Aid" may be applied :—

9379. (R. W. D. Christie, M.A.)—An annuity of £20 a year for 21 years is sold for £220; what rate per cent. did he get for his money?

Taking the ordinary formula, $PR^n = A(R^n-1)/(R-1)$, in which A = annual payment, n = number of years, R = amount to which £1 increases in one year at the particular rate of interest, P = present value; we have $220R^{21} = 20(R^{21}-1)/(R-1)$; and, by reduction,

$$11R^{22}-12R^{21}+1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a).$$

By the method of transformation already described, we obtain

$$\log\{K(\log 12-\log 11-\log x)-1\}+\log 11+22\log x=0,$$

in which K signifies the number corresponding to the logarithm filling the bracket next following. We now proceed by successive trials, and approach to the correct value of $\log x$, by correcting according to previous errors. We make a start by observing, from the log. within brackets, that $\log x<(\log 12-\log 11)$.

Trial $\log x$.	K $(\log 12-\log 11-\log x)$.	$\log[K(\log 12-\log 11-\log x)-1]$.	$(\log 11+22\log x)$.	Error.
0·0350	1·C064	$\bar{3}$·806180	1·811393	$\bar{1}$·6176
0·0300	1·0181	$\bar{2}$·257679	1·701393	$\bar{1}$·9591
0·0294	1·0195	$\bar{2}$·290035	1·688193	$\bar{1}$·9782
0·0290	1·0204	$\bar{2}$·309630	1·679393	$\bar{1}$·9890
0·0286	1·0214	$\bar{2}$·330414	1·670593	0·0010

We now know that $\log x\ (\equiv \log R)$ is slightly greater than 0·0286, but this is near enough for practical purposes.

Therefore R = 1·0681, and the percentage required is 6·81.

[General O'Connell solves the question as follows, beginning with the same equation (α), but putting $R = 1{\cdot}0r$:—

With the aid of Thomson's slide-rule for powers and roots, it will easily be seen that $(1{\cdot}0r)^{21} = 4$ gives a good first approximation.

$$\begin{aligned}
\log(1{\cdot}0r)^{21} = \log 4 &= {\cdot}6020600\\
\tfrac{1}{21}\log(1{\cdot}0r)^{21} &= {\cdot}0286695\\
\log(1{\cdot}0r)^{22} &= {\cdot}6307295\\
\log{\cdot}9998987 &= \bar{1}{\cdot}9999560\\
&\qquad 22\\
22\log{\cdot}9998987 &= \bar{1}{\cdot}9990320\\
21\log{\cdot}9998987 &= \bar{1}{\cdot}9990760
\end{aligned}$$

$$\begin{array}{r|l} 2 & {\cdot}6297615 \\ 11 & {\cdot}31488075 \\ & {\cdot}02862552 \end{array}$$

$$= \log 1{\cdot}068134.$$

Therefore the rate of interest = 6·8134 per cent.

$$\begin{aligned}
\log 12 &= 1{\cdot}0791812\\
\log 11 &= 1{\cdot}0413927\\
\log\tfrac{12}{11} &= {\cdot}0377885\\
\log(1{\cdot}0r)^{21} &= {\cdot}6020600
\end{aligned}$$

(1) ..... $\log(1{\cdot}0r)^{22}$ =	·6307295	·6398485
	$\bar{1}$·9990320	$\bar{1}$·9990760
(2)	·6297615	·6389245

Below are the natural numbers corresponding to the above logarithms,

(1)	4·272967	4·363636
(2)	4·263453	4·354362

Below are the computations for the successive approximations,

(1) ..... $4{\cdot}272967 \times 22 = 94{\cdot}005$
$-4{\cdot}363636 \times 21 = 91{\cdot}636$
$2{\cdot}369$
$+ \cdot 090909$
$+ \cdot 000240$
$\frac{-{\cdot}00024}{2{\cdot}869} = -{\cdot}0001013.]$

(2) ..... $4{\cdot}263453$
$-4{\cdot}354362$
$+ \cdot 090909$
$\cdot 000000$

It is desirable, however, to warn the reader against employing the *Aid to Approximate Calculation* for processes to which it is in no way adapted. It is not applicable to cases in which a result is required consisting of more than five, or at most six, significant figures. Where lengthy results are looked for, as in the multiplication or division of large numbers, and an arithmometer is not at hand, considerable relief may be obtained by the following methods, which have the advantage of evolving the figures of the result, whether product or quotient, in the order of their importance.

(D.) *How to Multiply when the numbers are large.*

Let A, B, C, &c., represent the figures of the multiplier, and a, b, c, &c., those of the multiplicand. Then

$$(A, B, C, \ldots)(a, b, c, \ldots) = Aa + (Ab + Ba) + (Ac + Bb + Ca) + \&c.$$

And the several terms may be represented by the successive vertical columns in the following diamond-shaped series :—

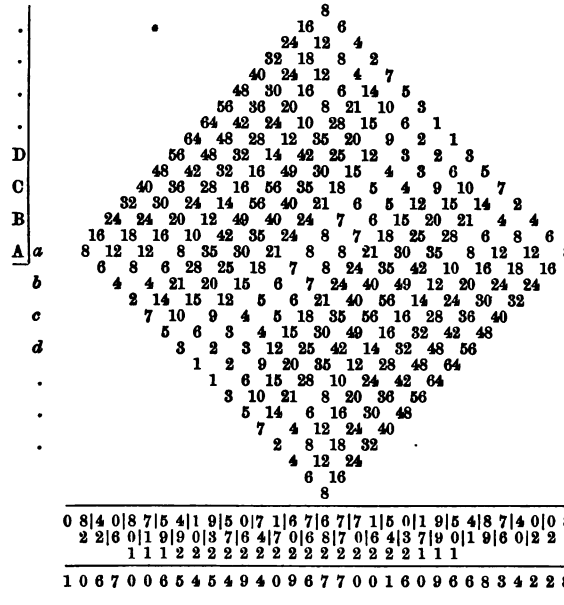
$$\begin{array}{ccccccccc}
 & & & & Ae & & & & \\
 & & & Ad & & Be & & & \\
 & & Ac & & Bd & & Ce & & \\
 & Ab & & Bc & & Cd & & De & \\
Aa & & Bb & & Cc & & Dd & & Ee \\
 & Ba & & Cb & & Dc & & Ed & \\
 & & Ca & & Db & & Ec & & \\
 & & & Da & & Eb & & & \\
 & & & & Ea & & & &
\end{array}$$

Such being the case, if we write the figures of the multiplicand in a column near the left margin of the page, in their proper order, and the figures of the multiplier on a slip of paper, in their reverse order, and move the slip alongside the page one figure at a time, we obtain a ready means of seeing what factors enter into each term of the result; and summing the minor terms or products in a double row at the foot, with a third row for occasional figures carried, we are able to write down the result figure by figure from the commencement.

The following instance is taken from the pamphlet describing Tate's *Improved Arithmometer*, and is there given as a case needing to be broken up into four parts in order to come within the scope of the instrument. The method here adopted enables us to give the result true to any number of figures to which we may feel disposed to carry it, since we can stop at any column, and be correct to, at least, the column next but one preceding. After some practice also, there is no need to record each separate product in the columns, but only the sum at the foot. The double row for enter-

ing the several sums renders it easy to detect and rectify any error that may occur. (See example M.)

Multiply 1234567887654321
by 8642753113572468 (M)



0 8|4 0|8 7|5 4|1 9|5 0|7 1|6 7|6 7|7 1|5 0|1 9|5 4|8 7|4 0|0 8
2 2|6 0|1 9|9 0|3 7|6 4|7 0|6 8|7 0|6 4|3 7|9 0|1 9|6 0|2 2
1 1 1 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1

1 0 6 7 0 0 6 5 4 5 4 9 4 0 9 6 7 7 0 0 1 6 0 9 6 6 8 3 4 2 2 8

(E.) *How to Divide when the numbers are large, or a lengthy quotient is required.*

The same principle will serve us in Division also, though requiring rather more wit in the application. Considering *a*, *b*, *c*, &c., of the quotient, as *x*, *y*, *z*, &c., until found, we take a slip of paper as in the former case, with the figures of the divisor on it in reverse order; and having found the first figure of the quotient in the usual way, we enter it near the top of the left-hand margin of the page. Placing A of the divisor (on the slip) alongside, we easily find A*a*. This we deduct from the first, or the first two figures of the dividend, as the case may be, and bring down a figure. Then moving the slip, until B of the divisor is opposite *a* of the quotient, we have B*a*, which we mentally deduct from the figures arrived at in the dividend. What is left gives us a rough guide to *b*, since A*b* is necessarily less. But it is requisite to make a trial of the *b* indicated, by moving the slip a step further by anticipation, until C is opposite *a*; for B*b* + C*a* cannot exceed the *remainder* that will be left when the next figure is brought down. When once the principle is grasped, however, the process is not a difficult one; moreover, it is identical in

kind at every stage. Having found b, we enter it under a, and subtract $Ab + Ba$ from the given *remainder*, these being the products of the pairs of figures in juxta-position on slip and margin. We then bring down the next figure of the dividend and proceed as before, for c; and so on. (See N.)

(F.) *How to extract the Square Root of numbers great and small.*

The same principle is applicable to finding the square root of a number, provided we treat both A, B, C, &c., and a, b, c, &c., as unknown quantities, until found. We begin with a blank slip and a blank margin; but, having found the first figure of the root in the usual way, we enter it on both, in the positions for A and a, and subtract the product (or square) from the original number as usual. Then, moving the slip one step, we consider what figure put for both b and B, would suit the *remainder* (supplemented by *one*, not two figures, as usual, brought down). But we must move the slip a further step by anticipation, as in division, in order to see that the new figure is not too large. The only difference, indeed, between extraction of the square root and division, by the method under consideration, consists in the double entry of the figures. (See Q.)

It is quite possible to extend the principle to finding the cube root, or roots of higher powers, by using two or more slips.

Divide 7854368 by 472.

2		7854368	(N)
7		4	
4	1	38	
	6	31	
	6	75	
	4	68	
	0	74	
	6	70	
	1	43	
	0	40	
	1	36	
	6	32	
	9	48	
	5	46	
nearly		20	
		19	
		10	
		6	
		40	
		31	
		90	
		80	
		100	
		95	
		50	
		53	

Answer: 16640·6101695 nearly.

Find the square root of 19.

5		19	(Q)
3	4	16	
4	3	30	
	5	24	
	8	60	
	8	49	
	9	110	
	8	94	
	9	160	
	5	137	
	0	230	
nearly		200	
		300	
		262	
		380	
		338	
		420	
		382	
		380	
		392	

Answer: 4·35889895 nearly.

APPENDIX II.

SOLUTIONS OF UNSOLVED QUESTIONS.

By R. F. Davis, M.A.

8293. (By Asparagus.)—From a point O on an ellipse (axes $2a$, $2b$) are drawn three chords OP, OQ, OR normal to the ellipse at P, Q, R; prove that $OP \cdot OQ \cdot OR = \frac{a^2b^2}{a^2-b^2} \times$ diameter of curvature at O.

Solution.

Let T (h, k) be an external point from which tangents TU, TV and normals TO, TP, TQ, TR are drawn to the ellipse. Then

$$TO \cdot TP \cdot TQ \cdot TR : TU \cdot TV = b^2h^2 + a^2k^2 : a^2 - b^2.$$

(See Milne's *Companion to Weekly Problem Papers*, Chap. xiv., p. 264.) As T moves along the normal TO up to O, U and V approach indefinitely near to and ultimately coincide with O; and TU, TV, being always proportional to parallel semi-diameters, become ultimately equal. By describing a circle touching the ellipse at U and passing through O, we see that

limit of $TU \cdot TV / TO$ = limit TU^2 / TO = diameter of curvature at O.

Also $b^2h^2 + a^2k^2 = a^2b^2$, ultimately. Hence $OP \cdot OQ \cdot OR$ = &c.

8343. (Chas. E. McVicker, M.A.)—If ρ be the radius of the circle which cuts orthogonally the escribed circles of any triangle, and ρ_1, ρ_2, ρ_3 the radii of the circles cutting similarly the inscribed and two of the escribed circles; prove that

$$(1)\ 4\rho^2 = \frac{(b+c)(c+a)(a+b)-abc}{a+b+c} = r^2 + \frac{r_1r_2r_3}{r},\ (2)\ \rho^2 + \rho_1^2 + \rho_2^2 + \rho_3^2 = 4R^2.$$

Solution.

Let I, I_1, I_2, I_3 be the centres of the inscribed and escribed circles of a triangle ABC; i, i_1, i_2, i_3 those of the triangle abc formed by the mid-points of the sides. Then a is a point on the radical axis of the circles I_2, I_3, for the tangent from it to either circle $= \frac{1}{2}(b+c)$. The radical axis

is, therefore, a line through a perpendicular to the line of centres I_2I_3, or parallel to II_1, and therefore coincides with the bisector of the angle a. Thus i is the radical centre of the three escribed circles, and ρ = tangent from i to any one of them.

The perpendicular from D on I_2I_3 = semi-sum of perpendiculars from B, C = $\frac{1}{2}(b+c)\cos\frac{1}{2}A$.

The perp. from i on $I_2I_3 = \frac{1}{2}(b+c)\cos\frac{1}{2}A - \frac{1}{2}(s-a)\sec\frac{1}{2}A$

$$= \tfrac{1}{2}\sec\tfrac{1}{2}A\left\{(b+c)\cos A + a\right\}, \text{ for } ai = \tfrac{1}{2}Ai.$$

Therefore

$$\left\{\tfrac{1}{2}(b+c)\right\}^2 - \rho^2 = \tfrac{1}{4}(b+c)^2\cos^2\tfrac{1}{2}A - \tfrac{1}{16}\sec^2\tfrac{1}{2}A\left\{(b+c)\cos A + a\right\}^2$$

$$16\rho^2\cos^2\tfrac{1}{2}A = 4(b+c)^2\sin^2\tfrac{1}{2}A\cos^2\tfrac{1}{2}A + \left\{(b+c)\cos A + a\right\}^2$$

$$= (b+c)^2 + 2a(b+c)\cos A + a^2$$

$$= (b+c-a)^2 + 4a(b+c)\cos^2\tfrac{1}{2}A.$$

But $4r^2\cos^2\frac{1}{2}A = (b+c-a)^2\sin^2\frac{1}{2}A$, for $r = (s-a)\tan\frac{1}{2}A$;

therefore $4(4\rho^2 - r^2) = (b+c-a)^2 + 4a(b+c) = (a+b+c)^2$,

$$4\rho^2 = r^2 + s^2 = r^2 + r_1r_2r_3/r_1.$$

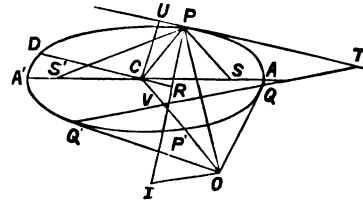
Similarly, i_1 is the radical centre of the circles I, I_2, I_3, and $4\rho_1^2 = r_1^2 + (s-a)^2$. Also

$$4(\rho^2 + \rho_1^2 + \rho_2^2 + \rho_3^2) = r^2 + r_1^2 + r_2^2 + r_3^2 + s^2 + (s-a)^2 + (s-b)^2 + (s-c)^2$$

$$= \ldots\ldots\ldots\ldots\ldots + a^2 + b^2 + c^2 = 16R^2.$$

8419. (Asparagus.)—At any point P of a given ellipse is drawn the radius of curvature PI, and a circle described on PI as diameter; this circle meets the ellipse again in Q, Q′: prove that the tangents to the ellipse at Q, Q′ meet in a point O on the circle; that the locus of O is the inverse of the ellipse with respect to the centre; and the first negative pedal of the locus of O is an ellipse similar to the given ellipse.

*Solution.**



Let P, P′ be two symmetrical points of an ellipse upon opposite sides of the major axis, so that CP, CP′ are equal to one another and equally inclined to CA. Upon CP′ produced take a point O, such that

$$CO \cdot CP = CS^2,$$

and from O draw tangents OQ, OQ′ to the ellipse. Let QQ′ meet the tangent at P in T, and CP′ in V.

Then I have shown in the *Reprint*, Vol. xlv., Appendix iii., 10, that the circumcircle of OQQ′ passes through P, and OP is the Symmedian line through O. Moreover, since TP, TQQ′ are equally inclined to the axis, $TP^2 = TQ \cdot TQ'$, and TP touches the circle at P. Draw OI per-

* In the following solution it has been found necessary to change slightly the order of the question.

pendicular to OP, so that I is the extremity of the diameter of the circle through P. It is also shown in Appendix III., 12, that O lies upon the circumcircle of the triangle SPS′, PO being the Symmedian line of the triangle through P. From the similar triangles POS, PS′C,

$$PO \cdot CP = SP \cdot S'P = CD^2; \quad \text{also} \quad PI : PO = CP : PR,$$

hence $PI = CD^2/PR$, and I is the centre of curvature of P.

[This suggests the following simple construction for the centre of curvature at any point P of an ellipse:—Draw a circle around SPS′, and PO the Symmedian line, meeting the circle again in O. Then the point I, taken on the normal at P, so that OI is perpendicular to OP, will be the centre of curvature at P.]

By hypothesis, $CP' \cdot CO = CS^2$, hence the locus of O is the inverse of the ellipse with respect to its centre.

If CD be conjugate to CP, and CU perpendicular on PT, $CD \cdot CU$ = constant. Hence the first negative pedal of the inverse of an ellipse with respect to its centre is a similar and concentric ellipse turned through a right angle about the common centre.

[If α be the excentric angle of P, the equation to the circle OQQ′ is

$$x^2 + y^2 - (x/a)\cos\alpha\,(a^2 + c^2\cos^2\alpha) + (y/b)\sin\alpha\,(a^2\sin^2\alpha - c^2) + c^2(\cos^2\alpha - \sin^2\alpha) = 0.]$$

8582. (D. Edwardes.)—If SY, SZ, S′Y′, S′Z′ are the focal perpendiculars on tangents to an ellipse from a point O; prove that (1) YZ, Y′Z′, and a perpendicular from O on major axis are concurrent; and (2) if YZ meet OS′ in L, and Y′Z′ meet OS in M, the points L, M, S, S′ are concyclic.

Solution.

(1) The common chords of the auxiliary circle and the circles described upon OS, OS′ as diameters are concurrent; that is, YZ, Y′Z′ intersect on the perpendicular from O on the major axis.

(2) Since OS, OS′ are perpendicular to Y′Z′, YZ respectively, by similar triangles, $OL : OM = OZ : OY' = OS : OS'$, or $OL \cdot OS' = OM \cdot OS$.

8612. (Maurice d'Ocagne.)—Soient d et d' les cordes réelles communes à une conique C et à un point O considéré comme cercle de rayon nul. Du point M, pris sur l'une ou l'autre des droites d et d', menons à la conique C les tangentes MT et MT′, dont les points de contact sont T et T′, et prenons sur ces tangentes les points N et N′ tels que les angles TON et T′ON′ soient droits. La droite NN′ coupe en I la polaire du point O par rapport aux tangentes MT et MT′. Démontrer que la droite IO passe par le point de rencontre des droites d et d'.

Solution.

The envelope of a straight line which moves in such a manner that the product of the perpendiculars upon it from two fixed points S, S′ is constant, is a conic Σ having these points for foci. Reciprocating with respect to any fixed point O, the locus of a point P which moves in such a manner that OP^2 varies as the product of its perpendicular distances upon two fixed straight lines d, d' (the reciprocals of S, S′) is a conic Σ' (the reciprocal of Σ). This involves the well-known property that the perpendicular from A on the polar of B with respect to a circle (centre O) is to the perpendicular from B on the polar of A = OA : OB.

Since the equation to Σ' will be of the form

$$x^2+y^2=(lx+my+n)(l'x+m'y+n'),$$

the lines d, d' are to be regarded as the real common chords of the conic Σ' with a point-circle at O.

The given question is found by reciprocating the theorem: the line joining the intersections of the normals at the extremities of any focal chord of a conic with the middle point of that chord is parallel to the axis.

8684. (Professor Mahendra Nath Ray, M.A., LL.B.)—An indefinite straight line envelopes a fixed parabola whose equation, when referred to the axes of a given series of confocal ellipses, is $(x)^{\frac{1}{2}}+(y)^{\frac{1}{2}}=$ constant. Show that the locus of the poles of the straight line with respect to the confocal ellipses always passes through a fixed point.

Solution.

If O be an external point from which tangents OQ, OR are drawn to any one of a series of confocal conics, the envelope of QR is a certain parabola, whose focus F is found by making the $\angle FCS = \angle OCS$, and taking $CF \cdot CO = CS^2$. The directrix of the parabola is OC, and the axes of the ellipse are tangents (see Vol. xlv., Appendix iii., Art. 8, p. 170).

Conversely, if QR be any tangent to the above parabola, the straight line which is the locus of its poles with respect to the confocals, must pass through O.

8698. (Asparagus.)—The circle of curvature to the ellipse $x^2/a^2+y^2/b^2=1$ at a point P is drawn, and QQ′ is the common tangent to the ellipse and circle; prove that, if e be the eccentricity of the ellipse, and θ the excentric angle of the point P, then

$$QQ'=\frac{8ae^2\sin^3\theta\cos^3\theta(1-e^2\cos^2\theta)^{\frac{1}{2}}}{1-2e^2\cos^4\theta(1+2\sin^2\theta)+e^4\cos^6\theta(1+3\sin^2\theta)}.$$

8470. (Professor Wolstenholme, M.A., Sc.D.)—The circle of curvature is drawn to an ellipse at a point P whose focal distances are as $a^{\frac{1}{2}}+(a-b)^{\frac{1}{2}} : a^{\frac{1}{2}}-(a-b)^{\frac{1}{2}}$; PR is the common chord of the ellipse and circle, and meets QQ′, the common tangent, in the point O; prove that $OP=a+b$, $OQ=OQ'=a-b$, $2a$, $2b$ being the lengths of the axes of the ellipse.

Solution.

(8698). Let T (h, k) be the point of intersection of the common tangents TP, TQQ′ to the ellipse and its circle of curvature at P. Then T lies on the confocal hyperbola passing through P (TAYLOR'S *Geometry of Conics*, Chapter VIII., Ex. 644), and is therefore determined by the equations

$$(h/a)\cos\theta + (k/b)\sin\theta = 1, \quad h^2\sec^2\theta - k^2\operatorname{cosec}^2\theta = c^2;$$

so that
$$h = -a\cos\theta\left\{1 - e^2\cos^2\theta\,(1+\sin^2\theta)\right\}/\mathrm{U},$$
$$k = b\sin\theta\left\{1 - e^2\cos^4\theta\right\}/\mathrm{U},$$

where $\mathrm{U} = 1 - 2\cos^2\theta + e^2\cos^4\theta$.

Now $\mathrm{TQ}' = \mathrm{TP} = 2a\cos\theta\sin\theta\,(1-e^2\cos^2\theta)^{\frac{1}{2}}/\mathrm{U}$. From the theorem for the product of two tangents from an external point to an ellipse (MILNE'S *Companion*, Chap. XIV.),

$$\mathrm{TP}.\mathrm{TQ} : \mathrm{ST}.\mathrm{S'T} = (h^2/a^2 + k^2/b^2 - 1) : (h^2/a^2 + k^2/b^2),$$

we shall, by substituting for the coordinates of T, obtain the value of TQ.

It is thus found $h^2/a^2 + k^2/b^2 = \mathrm{D}/\mathrm{U}^2$, where D = denominator of question, and $h^2/a^2 + k^2/b^2 - 1 = 4\cos^2\theta\sin^2\theta\,(1-e^2\cos^2\theta)^2/\mathrm{U}^2$,

$\mathrm{ST}.\mathrm{S'T} = -$(square of semi-diameter of hyperbola conjugate to CT, which makes an angle $\tan^{-1}(h\tan^2\theta/k)$ with the axis of x)
$$= a^2\mathrm{H}\,(1-e^2\cos^2\theta)/\mathrm{U}^2,$$

where
$$\mathrm{H} = 1 - e^2\cos^2\theta\,(4 - 6\cos^2\theta + 4\cos^4\theta) + e^4\cos^8\theta.$$

Thus $\mathrm{TP}.\mathrm{TQ} = 4a^2\cos^2\theta\sin^2\theta\,(1-e^2\cos^2\theta)^3.\,\mathrm{H}/\mathrm{U}^2\mathrm{D}$;

and $\mathrm{TQ}/\mathrm{TQ}' = \mathrm{H}/\mathrm{D}$, $\mathrm{QQ}'/\mathrm{TQ}' = (\mathrm{D}-\mathrm{H})/\mathrm{D} = 4e^2\cos^2\theta\sin^2\theta.\,\mathrm{U}/\mathrm{D}$,

$$\mathrm{QQ}' = 8ae^2\cos^3\theta\sin^3\theta\,(1-e^2\cos^2\theta)^{\frac{1}{2}}/\mathrm{D}.$$

(8470). When $a^2\sin^4\theta = b^2\cos^4\theta$, U = 0, and T is at infinity. Hence PQ′ is the diameter of curvature, and $\{\mathrm{TQOQ}'\}$ being harmonic, O bisects QQ′. If we put $\sin^2\theta = b/(a+b)$, and $\cos^2\theta = a/(a+b)$ in the above expression for QQ′, it becomes $= 2(a-b)$. Moreover

$$2(\mathrm{OP}^2 + \mathrm{OQ}^2) = \mathrm{PQ}^2 + \mathrm{PQ}'^2 = 4(\mathrm{CP}^2 + \rho^2) = 4(a^2 + b^2),$$

and $\mathrm{OP} = a + b$; for $\mathrm{CD}^2 = ab = \rho^2$.

[According to Professor WOLSTENHOLME, this value of QQ′ is a maximum for different positions of P on the ellipse.]

8707 & 8773. (Professor MUKHOPÂDHYÂY, B.A., F.R.S.E.)—

(8707). Show that the reciprocal polar of the evolute of the evolute of the conic $x^2/a^2 + y^2/b^2 = 1$, with respect to the circle described on the line joining the foci as diameter, is the curve $a^2x^2 + b^2y^2 = (a^2x^4 - b^2y^4)^{\frac{1}{2}}$.

(8773). Show that the reciprocal polar of the evolute of the evolute of the parabola $y^2 = 4a(x+2a)$, with respect to a circle of radius a, is the curve $y^2(3x^2 + 2y^2) = ax^3$.

Solution.

If P be any point on a curve, I the centre of curvature at P, then a straight line through I parallel to the tangent at P will touch the evolute

of the evolute, and the locus of the pole of this straight line with respect to the auxiliary circle will be the curve required.

(8773). Coordinates of P, $a(t^2-2)$, $2at$; I, $3at^2$, $-2at^3$. Identifying the equation $t(y+2at^3) = x-3at^2$ with $xh+yk=a^2$, we have

$$h : k : 1 = 1 : -t : 3t^2+2t^4. \quad \text{Whence } k^2(3h^2+2k^2) = ah^3.$$

(8707). Coordinates of P, $a\cos a$, $b\sin a$; I, $(c^2/a)\cos^3 a$, $-(c^2/b)\sin^3 a$. Identifying the equation $x\cos a/a + y\sin a/b = c^2(\cos^4 a/a^2 - \sin^4 a/b^2)$ with $xh+yk=c^2$, we have $\cos^3 a : \sin^3 a : 1 = a^3h^3 : b^3k^3 : a^2h^4-b^2k^4$. Hence we have $a^2h^2+b^2k^2 = (a^2h^4-b^2k^4)^{\frac{1}{2}}$.

8734. (Maurice d'Ocagne.)—On donne deux points F et P et une droite Δ parallèle à FP. Si du point P on mène des tangentes t et t' à une parabole variable π, de foyer F, tangente à Δ, 1° la polaire du point F par rapport aux droites t et t' est une droite fixe δ; 2° la polaire du point P par rapport à la parabole π passe par un point fixe situé sur δ.

Solution.

Draw FY perpendicular to the tangent Δ, and produce it to Z, so that FY = YZ; then the directrix of every parabola passes through Z. Since FP is perpendicular to FZ, the polar of Z with respect to π is fixed, and coincides with FP, which passes through P. Hence the polar of P with respect to π passes through Z, and PZ is the fourth harmonic to t, PF, t'.

8776. (Asparagus.)—From a point O are drawn four normals OP, OQ, OP′, OQ′ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and OP, OQ make complementary angles with the axes; prove that (1) the locus of O is the curve

$$(x^2-y^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2 = (a^2-b^2)\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2;$$

(2) the envelope of PQ is the hyperbola $\left(\frac{x^2}{a^4}-\frac{y^2}{b^4}\right)(a^2-b^2)=1$;

and (3) P′Q′ will be normal to the rectangular hyperbola $4(x^2-y^2)=a^2-b^2$.

Solution.

Since $y = mx + (a^2m^2+b^2)^{\frac{1}{2}}$ is the equation to any tangent, the directions of the tangents TP, TQ that can be drawn through T (X, Y) are determined by the equation $m^2(X^2-a^2) - 2m\,XY + Y^2 - b^2 = 0$;

the product of whose roots must = 1, if TP, TQ are inclined at complementary angles to either axis. Hence $X^2-Y^2 = a^2-b^2 = c^2$, or T lies on a concentric rectangular hyperbola. Put $X = c\cosh\phi$, $Y = c\sinh\phi$; then the equation of PQ, the polar of T, will be

$$xc\cosh\phi/a^2 + yc\sinh\phi/b^2 = 1,$$

and (2) PQ envelopes the concentric hyperbola whose semi-axes are a^2/c, b^2/c. The equation to P'Q' is derived from that of PQ (WOLSTENHOLME'S *Mathematical Problems*, p. 156), $x \operatorname{sech} \phi + y \operatorname{cosech} \phi = c$; it is therefore normal (3) to the rectangular hyperbola whose semi-axes are each $= \frac{1}{2}c$.

Similarly, since $y = mx - c^2m\,(b^2m^2 + a^2)^{-\frac{1}{2}}$ is the equation to any normal, the directions of the normals OP, OQ, OR, OS that can be drawn through O (h, k) are determined by the equation

$$b^2h^2m^4 - 2b^2hkm^3 + (b^2k^2 + a^2h^2 - c^4)\,m^2 - 2a^2hkm + a^2k^2 = 0,$$

the product of two of the roots of which must $= 1$. Hence m also satisfies the reciprocal equation

$$a^2k^2m^4 - 2a^2hkm^3 + (\ldots)\,m^2 - 2b^2hkm + b^2h^2 = 0.$$

Subtracting and adding these two equations, we get

$$m/(m^2 + 1) + (b^2h^2 - a^2k^2)/2c^2hk = 0,$$

and $(b^2h^2 + a^2k^2)\,(m^4 + 1) - 2m\,(m^2 + 1)\,(a^2 + b^2)\,hk + 2m^2\,(b^2k^2 + a^2h^2 - c^4) = 0.$

Hence, by substitution, we obtain

$$\left\{4c^4h^2k^2 - 2\,(b^2h^2 - a^2k^2)^2\right\}(b^2h^2 + a^2k^2) + 4c^2h^2k^2\,(a^2 + b^2)\,(b^2h^2 - a^2k^2) + 2\,(b^2h^2 - a^2k^2)^2\,(b^2k^2 + a^2h^2 - c^4) = 0,$$

or

$$c^2\,(b^2h^2 - a^2k^2)^2 = (h^2 - k^2)\,(b^2h^2 + a^2k^2)^2.$$

8785. (MAURICE D'OCAGNE.)—Si les points A et B se déplacent sur la directrice D d'une conique C, dont F est le foyer correspondant, de telle façon que l'angle AFB soit constamment droit, le quadrilatère formé par les tangentes menées des points A et B à la conique C, reste inscrit dans une conique ayant le point F pour foyer et la droite D pour directrice correspondante.

Solution.

Reciprocate the theorem: the sides of the quadrilateral formed by the extremities of any two perpendicular diameters of a circle envelope a concentric circle.

8791. Professor GENESE, M.A.)—If a conic circumscribes a parallelogram OABC, and a variable straight line through O meets the opposite sides in P, Q, and the curve in R; prove that

PR : RQ varies as $OP^2 : OQ^2$, or as AP : CQ.

Solution.

Let the straight line meet BC in Q, the curve in R, and AB produced in P. Then the ratio compounded of PR . OP : AP . BP and BQ . CQ : OQ . RQ

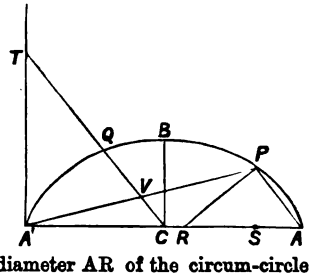
is constant. Hence

$$PR : RQ \propto AP . BP . OQ : BQ . CQ . OP.$$

But by similar triangles $OQ : CQ = OP : OA$, $OP . CQ = OQ . OA$; therefore $PR : RQ \propto AP . BP : BQ . OA$, or $AP^2 : OA^2$, or $AP^2 : AB^2$, or $OP^2 : OQ^2$; and also $\propto AP : OA . OC/AP$, or $AP : CQ$.

8809. (J. J. Walker, M.A., F.R.S.)—Show that the time of sliding down any chord of a central conic, with axis vertical, to the lowest vertex, is equal to the time of falling down a height equal to the parallel focal chord.

Solution.



Let the axis AA′ be vertical, A being the lowest vertex. Then, if AP be any chord, draw PR, CVQT perpendicular and parallel to AP.

Then, by similar triangles,

$$AR : AP = CT : A'C;$$

$$\therefore AR . AC = AP . CT = 2CV . CT = 2CQ^2,$$

But, if pSp' be a focal chord parallel to AP, $pSp' : AA' = CQ^2 : AC^2$.

Thus $AR = pSp'$; and the time of sliding down the chord PA is equal to the time of falling vertically down the diameter AR of the circum-circle of APR, &c.

8828. (Professor Nash, M.A.)—Prove that the envelope of the polar of any point with respect to a series of Tucker circles is a parabola whose axis is perpendicular to the line of centres of the circles.

Solution.

Let ACA′, BCB′ be the axes and Ω, Ω' the foci of the Brocard ellipse. Then, if any point O $(0, -\beta)$ be taken on the minor axis, the circle described from O as centre, with radius $= O\Omega/\epsilon$, will have double contact with this ellipse, and is therefore a Tucker circle. Its equation will be

$$x^2 + (y+\beta)^2 = (a^2\epsilon^2 + \beta^2)/\epsilon^2, \quad \text{or} \quad x^2 + y^2 + 2\beta y - a^2 - b^2\beta^2/a^2\epsilon^2 = 0.$$

The polar of a fixed point (h, k) will be

$$xh + yk + \beta(y+k) - a^2 - b^2\beta^2/a^2\epsilon^2 = 0,$$

and therefore envelopes the parabola

$$a^2\epsilon^2(y+k)^2 + b^2(xh + yk - a^2) = 0.$$

A particular case may be noted. The envelope of the polar of Ω is a parabola having Ω for its focus and the Ω-directrix for tangent at the vertex.

8836. (Professor WOLSTENHOLME, M.A., Sc.D.)—A chord of the parabola $y^2 = 4ax$ is drawn through the focus S and through a point P $[2am, a(1+m^2)]$ on the parabola $y^2 = 4a(x-a)$; prove that, (1) of all chords of the former parabola which can be drawn through P, the chord aforesaid is always either a maximum or a minimum; a maximum if SP $> 6a$, and a minimum if SP $< 6a$; (2) when this chord is a maximum there are also two real minimum chords, each of the same length $4am$, and inclined to the axis at complementary angles; the length of the maximum chord being $a(4+m^2)$; and (3) find the envelope of the straight lines on which the minimum chords lie.

Solution.

The equation determining the segments of a chord of the outer parabola drawn through P so as to make an angle θ with the axis of x is

$$\{r\sin\theta + 2am\}^2 = 4a\{r\cos\theta + a(1+m^2)\},$$

or
$$r^2\sin^2\theta - 4ar(\cos\theta - m\sin\theta) - 4a^2 = 0.$$

Hence (length)^2 of cord $= (r_1+r_2)^2 - 4r_1r_2$

$$= 16a^2(\cot^2\theta+1)\{(\cot\theta - m)^2 + 1\}.$$

Differentiating with respect to $\cot\theta$, the length of the cord will be maximum or minimum when

$$(2\cot\theta - m)(\cot^2\theta - m\cot\theta + 1) = 0.$$

This has always one real root $\cot\theta = \frac{1}{2}m$, so that the chord

$$\tfrac{1}{2}m(y-2am) = x - a(1+m^2), \text{ or } my = 2(x-a),$$

is always a maximum or minimum. The second differential coefficient with respect to $\cot\omega$ will be found for this value to be positive or negative according as $m^2 <$ or > 4; that is, according as SP, which $= a + a(1+m^2)$, is $<$ or $> 6a$. The chord is a minimum in the first case, and a maximum in the second.

When $m^2 > 4$, and SP $> 6a$, the equation

$$\cot^2\theta - m\cot\theta + 1 = 0$$

has real roots, giving two directions at complementary angles to the axis; these are evidently minimum values separating the two maximum values, the focal chord through P and the diameter through P (for which $\cot\theta = \infty$).

On substituting for $\cot\theta$, in the expression for the length of the chord, the values of the question $4am$ and $a(4+m^2)$, are easily obtained.

The equation to the two minimum chords is

$$\{x - a(1+m^2)\}^2 - m[\{x-a(1+m^2)\}(y-2am)] + (y-2am)^2 = 0,$$

or
$$a^2m^4 - aym^3 - 4a^2m^2 + my(x+3a) - (x-a)^2 - y^2 = 0.$$

Writing $x-a$ for x, which is transferring to parallel axes through the focus, the envelope is found to be

$$\{x(y^2-4ax)/4a^3 + \tfrac{4}{3}\}^3 = 27\{(y^2-4ax)(3y^2-8ax)/48a^4 + \tfrac{8}{27}\}^2,$$

which separates into two factors,

$$(y^2-4ax)^2 = 0, \text{ and } 27ay^4 - 4y^2(x^2+36a^2x) + 16a(x^2+4a^2)^2 = 0,$$

or
$$4(x^2-12a^2)^3 = (2x^3 - 27ay^2 + 72a^2x)^2,$$

a curve having two real and two impossible cusps when $x^2 = 12a^2$, $9y^2 = 32ax$; and two impossible nodes when $y = 0$, $x^2 + 4a^2 = 0$.

8983. (Professor MATHEWS, M.A.)—ABCD is a convex quadrilateral of which the sides touch one circle, while the vertices lie on another. Tangents are drawn to the circum-circle at A, B, C, D so as to form another convex quadrilateral; prove that the area of the latter

$$= 2r^2 \frac{(s\sigma - 2abcd)(abcd)^{\frac{1}{2}}\sigma}{s(\sigma - bcd)(\sigma - cda)(\sigma - dab)(\sigma - abc)},$$

where r is the radius of the circle ABCD; a, b, c, d are the lengths of AB, BC, CD, DA, respectively;

$$2s = a+b+c+d, \quad 2\sigma = bcd + cda + dab + abc.$$

Solution.

Area of the quadrilateral ABCD

$$= \{(s-a)(s-b)(s-c)(s-d)\}^{\frac{1}{2}} = (abcd)^{\frac{1}{2}}, \text{ since } a+c = b+d = s.$$

Area of the quadrilateral A′B′C′D′ formed by the tangents $= r$ (semi-perimeter) $= r^2(\tan\alpha + \tan\beta + \tan\gamma + \tan\delta)$,

where $\alpha, \beta, \gamma, \delta$ are the angles subtended at the circumference by a, b, c, d respectively, so that $\alpha + \beta + \gamma + \delta = \pi$,

$$= r^2 \sin B(\sec\alpha \sec\beta + \sec\gamma \sec\delta).$$

Now, $\cos\alpha\cos\beta = \sin\alpha\sin\beta + \cos(\alpha+\beta)$;

and $\cos\gamma\cos\delta = \sin\gamma\sin\delta + \cos(\gamma+\delta)$;

therefore $\cos\alpha\cos\beta + \cos\gamma\cos\delta = \sin\alpha\sin\beta + \sin\gamma\sin\delta = (ab+cd) / 4r^2$.

Also $(ab+cd)^2 . \sin^2 B = (s-a)(s-b)(s-c)(s-d) = abcd$.

Thus area A′B′C′D′ $= \frac{1}{4}(abcd)^{\frac{1}{2}} . \sec\alpha \sec\beta \sec\gamma \sec\delta$,

and it remains to express $\cos\alpha\cos\beta\cos\gamma\cos\delta$ in terms of the sides.

From the triangle ABC we get

$$2b . AC . \cos\alpha = b^2 + AC^2 - a^2 = b^2 + \{(ac+bd)(ad+bc) / (ab+cd)\} - a;$$

therefore $2(ab+cd) . AC . \cos\alpha = a(b^2+c^2+d^2-a^2) + 2bcd$

$$= 2(bcd + cda - dab + abc) = 4(\sigma - dab);$$

for $(b-c)^2 = (a-d)^2$ and $b^2+c^2+d^2-a^2 = 2bc - 2ad + 2d^2 = 2\{be + d(c-b)\}$.

Similarly for $\cos\beta$, &c.:

and $\cos\alpha\cos\beta\cos\gamma\cos\delta = 16(\sigma - bda) \ldots\ldots / \{(ab+cd)(ac+bd)(ad+bc)\}$

$$= \ldots\ldots\ldots\ldots / \{16r^2 . abcd\}^2.$$

Thus area A′B′C′D′ $= 8r^4(abcd)^{\frac{1}{2}} / (\sigma - bcd) \ldots\ldots$

Since $2\sigma = ac(b+d) + bd(a+c) = s(ac+bd)$, $ac+bd = 2\sigma / s$.

Also $(ab+cd)(ad+bc) = bd(a^2+c^2) + ac(b^2+d^2)$

$$= (ac+bd)s^2 - 4abcd = 2(s\sigma - 2abcd).$$

Hence $16r^2 = (ab+cd)(ac+bd)(ad+bc) / (s-a) \ldots\ldots$

$$= \sigma(s\sigma - 2abcd) / 4s . abcd;$$

therefore Professor MATHEWS' result can be further simplified, and will be found to agree with that given above.

9242. (Professor SCHOUTE.)—The triangle ABC is the intersection of the plane δ with the faces of the trihedron $\alpha\beta\gamma$. To find the locus of the centroid of ABC, when δ moves about (1) a fixed line, (2) a fixed point.

Solution.

Take the faces of the trihedron as coordinate planes and their intersections as coordinate axes. Then, if $x/\alpha + y/\beta + z/\gamma = 1$ be the equation to the plane δ, and X, Y, Z the coordinates of the centroid of the triangle ABC, $3X = \alpha$, $3Y = \beta$, $3Z = \gamma$.

Therefore (2) if δ moves about a fixed point x', y', z',

$$x'/X + y'/Y + z'/Z = 3$$

represents the surface locus of X, Y, Z.

(1) If $Ax + By + Cz - 1 + \lambda(A'x + B'y + C'z - 1) = 0$ (where λ is variable) be the equation to δ, so that δ moves about a fixed line; putting $y = z = 0$, we have $\alpha = (3X) = (\lambda + 1) / (A + \lambda A')$.

Hence the curve locus of X, Y, Z is found (by eliminating λ) to be

$$(1 - 3AX) / (1 - 3A'X) = \ldots\ldots = \ldots\ldots \quad \text{[which is a twisted cubic.]}$$

9249. (D. Edwardes.)—The normal at a point P of an ellipse meets the curve again in Q. Find the maximum value of the angle CQP, C being the centre of the ellipse.

Solution.

Let $ax \sec\theta - by \operatorname{cosec}\theta = a^2 - b^2$ be the equation to the normal at the point whose excentric angle is θ. Then the equation

$$(ax \sec\theta - by \operatorname{cosec}\theta)^2 = (a^2 - b^2)^2 (x^2/a^2 + y^2/b^2),$$

being homogeneous and of the second degree in y/x, represents the lines joining the origin to the two points of intersection of the normal at P with the curve; one of which is CP, whose equation is $y/x = b \tan\theta/a$. The equation to CQ (the other of these lines) is therefore

$$\frac{y}{x} = \frac{b}{a} \cot\theta \, \frac{a^4 \tan^2\theta + 2a^2b^2 - b^4}{b^4 \cot^2\theta + 2a^2b^2 - a^4}.$$

Now PQ makes with an angle with the positive direction of the axis of x whose tangent is $a \tan\theta/b$.

If ϕ be the angle between CQ and PQ, it is found that

$$\cot\phi = \frac{2ab}{a^2 - b^2} \left\{ \frac{1}{\sin 2\theta} - \frac{(a^2 - b^2)^2 \cot\theta}{a^4 + b^4 \cot^2\theta} \right\}$$

since $\operatorname{cosec} 2\theta = (1 + \cot^2\theta) / 2 \cot\theta$; the above expression for $\cot\phi$ may be considered a function in which $\cot\theta$ is the variable. In order to find the maximum or minimum values of ϕ, we must differentiate with respect to $\cot\theta$ and put result = 0. Hence after reduction,

$$b^8 \cot^6\theta + \cot^4\theta (b^4 - 2a^2b^2)^2 - \cot^2\theta (a^4 - 2a^2b^2)^2 - a^8 = 0.$$

This equation being a cubic in $\cot^2\theta$ has certainly one real positive root, $\cot^2\theta = \cot^2\alpha$, suppose; and the four points, lying symmetrically in the four quadrants whose excentric angles are $\pm\alpha$, are the four points at which the angle CQP is a maximum, and they alternately separate the extremities of the axes at which the same angle is obviously a minimum.

9261. (Professor NASH, M.A.)—The general equation of a parabola in trilinear coordinates is

$$(l\alpha+m\beta+n\gamma)^2=(a\alpha+b\beta+c\gamma)(l'\alpha+m'\beta+n'\gamma) \text{ or } L^2=SL';$$

if $$A=\Sigma l^2-2\Sigma mn\cos A,\quad A'=\Sigma l'^2-2\Sigma m'n'\cos A,$$

$$B=\Sigma ll'-\Sigma(mn'+m'n)\cos A,$$

prove that the equation of the axis is $2AL=BS$, the focus is the intersection of this line with $4AL'=A'S$, the equation of the directrix is $4BL-4AL'=A'S$, and the Latus rectum $=2\Delta\,(AA'-B^2)^{\frac{1}{2}}/A^{\frac{3}{2}}$.

Solution.

Let θ be the angle between the lines L, L', then $\tan\theta=D/B$, where $D=\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ \sin A, & \sin B, & \sin C \end{vmatrix}$. It will be found that $D^2+B^2=AA'$, so that $\cos\theta=B/(AA')^{\frac{1}{2}}$; if B be supposed positive, θ is the acute angle between L, L'.

Let Q (α, β, γ) be any point on the parabola, QV the ordinate parallel to the tangent PT (L') to the diameter PV (L). Draw QD, VR perpendiculars on QV, PT respectively. Then $QD^2=4FY.VR$, where FY is the perpendicular from the focus F upon L', or

$$(l\alpha+m\beta+n\gamma)^2/A=4FY.(l'\alpha+m'\beta+n'\gamma)/A'^{\frac{3}{2}}.$$

But, since Q lies on the parabola $L^2=SL'$, hence $4FY=S.A'^{\frac{1}{2}}/A$. Now the axis is parallel to L at a perpendicular distance = ordinate of P $=2AY=2FY\cos\theta$, hence its equation is

$$L/A^{\frac{1}{2}}=\tfrac{1}{2}S\,(A'^{\frac{1}{2}}/A)\,(B/AA')^{\frac{1}{2}} \text{ or } 2AL=BS.$$

The focus lies on a straight line parallel to L' at a distance from it = FY, hence its equation is $L'/A'^{\frac{1}{2}}=\tfrac{1}{4}SA'^{\frac{1}{2}}/A$ or $4AL'=A'S$.

If PM be the perpendicular on the directrix, M and F are equally distant from L' (on opposite sides), hence the equation to the directrix is of the form $\lambda L=4AL'+A'S$; and at the point Z where L' meets the directrix the perpendicular ZM on $L=FZ=FY\sec\theta$; therefore putting $L'=0$, we have $\lambda FY.\sec\theta=A'S/A'^{\frac{1}{2}}$, whence $\lambda=4B$.

The latus rectum $=4FY\sin\theta=S\,(A'^{\frac{1}{2}}/A)(1-B^2/AA')^{\frac{1}{2}}=S\,(AA'-B^2)^{\frac{1}{2}}/A^{\frac{3}{2}}$.

9263 & 7822. (9263). (Prof. WOLSTENHOLME, Sc.D.)—Prove that, if $\alpha+\beta+\gamma+\delta=\pi$ (or any odd multiple of π),

$$\begin{aligned}
&4\,(\sin\beta\sin\gamma+\sin\alpha\sin\delta)(\sin\gamma\sin\alpha+\sin\beta\sin\delta)(\sin\alpha\sin\beta+\sin\gamma\sin\delta)\\
&=4\,(\cos\beta\cos\gamma+\cos\alpha\cos\delta)(\cos\gamma\cos\alpha+\cos\beta\cos\delta)(\cos\alpha\cos\beta+\cos\gamma\cos\delta)\\
&=(-\sin\alpha+\sin\beta+\sin\gamma+\sin\delta)(-\sin\beta+\sin\gamma+\sin\delta+\sin\alpha)(\ldots\ldots)(\ldots\ldots)\\
&=(-\cos\alpha+\cos\beta+\cos\gamma+\cos\delta)(-\cos\beta+\cos\gamma+\cos\delta+\cos\alpha)\,(\ldots\ldots)(\ldots\ldots)\\
&=(\sin\alpha\cos\alpha+\sin\beta\cos\beta+\sin\gamma\cos\gamma+\sin\delta\cos\delta)^2\\
&=4\sin(\beta+\gamma)\sin(\alpha+\delta)\sin(\gamma+\alpha)\sin(\beta+\delta)\sin(\alpha+\beta)\sin(\gamma+\delta).
\end{aligned}$$

(7822). (E. BUCK, M.A.)—If R be the radius of a circle round a quadrilateral whose sides are a, b, c, d, and $2s\equiv a+b+c+d$, prove that

$$(4R)^2=\left[(ab+cd)(ac+bd)(ad+bc)\right]/\left[(s-a)(s-b)(s-c)(s-d)\right].$$

Solution.

(9263). Denote the six expressions whose equality it is required to prove by the letters P, Q, R, S, T, V. Then, since $2\alpha+2\beta+2\gamma+2\delta=2\pi$, these angles may be taken as the angles subtended at the centre O of a circle (radius R) by the respective sides AB, BC, CD, DA (a, b, c, d), of an inscribed quadrilateral ABCD. Let ϕ be the angle between the diagonals AC, BD, and Δ the area of the quadrilateral.

(P). Since $bc+ad=2R\,.\,AC\sin\phi$, $ca+bd=AC\,.\,BD$,

and $ab+cd=2R\,.\,BD\sin\phi$,

therefore $(bc+ad)(ca+bd)(ab+cd)=4R^2\,.\,AC^2\,.\,BD^2\sin^2\phi=16\Delta^2R^2$.

Writing for a, $2R\sin\alpha$, &c., we have

$$P=64\Delta^2R^2/(4R^2)^3=\Delta^2/R^4.$$

(R). Since $(-a+b+c+d)(\ldots)(\ldots)(\ldots)=16\Delta^2$, hence the expression

$$R=16\Delta^2/(2R)^4=\Delta^2/R^4.$$

(T). Since $\frac{1}{2}R^2(\sin 2\alpha+\sin 2\beta+\sin 2\gamma+\sin 2\delta)=\Delta$, $\therefore\ T=\Delta^2/R^4$.

(V). Since $AC^2\,.\,BD^2\sin^2\phi=4\Delta^2$, and

$AC=2R\sin(\alpha+\beta)$ or $2R\sin(\gamma+\delta)$, $BD=2R\sin(\beta+\gamma)$ or $2R\sin(\alpha+\delta)$,

$\sin\phi=\sin(\gamma+\alpha)$ or $\sin(\beta+\delta)$, hence $V=\Delta^2/R^4$;

thus $$P=R=T=V=\Delta^2/R^4.$$

Moreover, since $(\pi-2\alpha)+(\pi-2\beta)+\ldots+\ldots=2\pi$, these formulæ also hold true when $\frac{1}{2}\pi-\alpha$, $\frac{1}{2}\pi-\beta$, $\frac{1}{2}\pi-\gamma$, $\frac{1}{2}\pi-\delta$ are written for α, β, γ, δ. This is tantamount to writing sine for cosine, and cosine for sine; and this interchange does not affect T.

Hence $Q=S=T=\Delta^2/R^4$, and all six expressions are equal.

[These formulæ are perhaps most easily proved by taking α, β, γ, δ as the roots of an equation of the form $\sin 2\theta+P\sin\theta+Q\cos\theta+R=0$.]

(7822). This theorem is an obvious consequence of one of the equations in Quest. 9263, since

$$a=2R\sin\alpha,\quad b=2R\sin\beta,\quad c=2R\sin\gamma,\quad d=2R\sin\delta.$$

9265. (Professor Mahendra Nath Ray, M.A., LL.B.)—If

$$\tan x_1\tan x_2\tan x_3\ldots\ldots\tan x_n=\tan A,$$

prove that $$\cos 2A=\frac{s_1+s_3+s_5+\ldots}{1+s_2+s_4+\ldots},$$

where s_r = the sum of the product of the quantities $\cos 2x_1$, $\cos 2x_2$, ... $\cos 2x_n$ taken r at a time.

Solution.

Squaring, we have $\tan^2x_1\tan^2x_2\tan^2x_3\ldots\ldots=\tan^2A$,

or $\{(1-\cos 2x_1)/(1+\cos 2x_1)\}\{(1-\cos 2x_2)/(1+\cos 2x_2)\}\ldots$

$=\{(1-\cos 2A)/(1+\cos 2A)\}$.

Hence $\cos 2A = \{(1+\cos 2x_1)(1+\cos 2x_2)\ldots-(1-\cos 2x_1)(1-\cos 2x_2)\ldots\}$
$+\{\quad\ldots\quad\ldots\quad\ldots\quad+\quad\ldots\quad\ldots\quad\ldots\quad\}.$
$= (s_1+s_3+s_5+\ldots)/(1+s_2+s_4+\ldots).$

9270. (Professor Satis Chandra Ray, M.A.)—If $p_1, p_2, p_3, p_4, p_5, p_6$ be the central perpendiculars on the tangent planes at the six points where the normals through a given point (X, Y, Z) meet the ellipsoid, and $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6$ be the normal distances of the point from the ellipsoid, show that

(1) $\Sigma p_1\rho_1 = 0$, (2) $p_1\rho_1 p_2\rho_2\ldots p_6\rho_6 = a^4b^4c^4\{1-X^2/a^2-Y^2/b^2-Z^2/c^2\}$.

Solution.

The equation to the normal at any point x, y, z of the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2 = 1$$

is $$a^2(X/x-1) = b^2(Y/y-1) = c^2(Z/z-1) \quad \ldots\ldots\ldots\ldots\text{(i.)},$$

where X, Y, Z are current coordinates.

The normals, therefore, that pass through a fixed point X, Y, Z, are the intersections of (1) with the ellipsoid.

Let each member of (1) $= t^2$, then

$$X/x = (t^2+a^2)/a^2, \quad \text{and} \quad x/a = aX/(t^2+a^2).$$

Hence $$a^2X^2(t^2+a^2)^{-2}+\ldots+\ldots = 0 \quad \ldots\ldots\ldots\ldots\text{(ii.)}$$

is the equation which determines t, which is of the 6th degree in t^2.

Now, if ρ be the distance of the point (x, y, z) from (X, Y, Z), we have

$$\rho^2 = (x-X)^2+\ldots+\ldots = t^4\{X^2(t^2+a^2)^{-2}+\ldots+\ldots\}.$$

Again, if p be the perpendicular from the centre on the tangent plane at (x, y, z), $1/p^2 = x^2/a^4+\ldots+\ldots = X^2(t^2+a^2)^{-2}+\ldots+\ldots$

Hence $p^2\rho^2 = t^4$; and substituting $p\rho$ for t^2 in (ii.), we get an equation of the 6th degree in $p\rho$, and

$$\Sigma p_1\rho_1 = 0, \quad p_1\rho_1 p_2\rho_2\ldots p_6\rho_6 = (1-X^2/a^2-Y^2/b^2-Z^2/c^2).$$

9283. (R. Knowles, B.A. Connected with Quest. 9109.)—A circle passes through two points P, Q on a rectangular hyperbola, and meets it again in C, D; prove that the perpendicular from the centre of the hyperbola on PQ bisects CD and passes through its pole with respect to the hyperbola, and *vice versâ*.

Solution.

Since the normal at any point of a rectangular hyperbola and the central radius-vector to the same point are equally inclined to the axes, it follows that, if two chords (such as PQ, CD) be equally inclined to the axes, the diameter perpendicular to one of the chords is conjugate to the other chord.

9284. (F. R. J. Hervey.)—Prove that the locus of the feet of the normals drawn from the centre of a circle to the variable conic passing through four points given on the circle is a rectangular hyperbola.

Solution.

The equation to the normal to the curve $f(x, y) = 0$ at the point (x, y) being

$$(X-x)\Big/\frac{df}{dx} = (Y-y)\Big/\frac{df}{dy},$$

where X, Y are current coordinates; the points at which the normal passes through the origin are given by the equation

$$y\frac{df}{dx} - x\frac{df}{dy} = 0.$$

Applying this to the curve

$$\lambda(x^2+y^2-a^2) + (Ax+By+C)(A'x+B'y+C') = 0,$$

which represents (if λ be variable) a variable conic passing through four fixed points on the circle, we get as the locus of the feet of the normals

$$(A'x+B'y+C')(Ay-Bx) + (Ax+By+C)(A'y-B'x) = 0,$$

or $(AB'+A'B)(y^2-x^2) + 2(AA'-BB')xy + (AC'+A'C)y - (BC'+B'C)x = 0,$

which represents a rectangular hyperbola passing through the centre of the circle and through the mid-points and intersection of the two chords $Ax+By+C=0$ and $A'x+B'y+C'=0$. As no restriction was made in the original equation as to which pair of chords joining the four given points should be selected, we conclude the rectangular hyperbola passes through the centre of the circle, the mid-points of the sides of the quadrilateral formed by the four given points, the mid-points of the two diagonals, and the vertices of the self-conjugate triangle.

[*Otherwise*:—We may write, for the variable conic, $u+\lambda u' = 0$ ... (1), which becomes a circle when $\lambda = a$ (suppose), and for the rectangular hyperbola (generally variable) which determines the feet of normals from a fixed point O, $v+\lambda v' = 0$ ... (2); giving, for the locus of intersection, $uv'-u'v = 0$ ... (3). But, if O be taken at the centre of the circle, the supposition $\lambda = a$ must make (2) true identically; whence (2) and (3) have, in this case, the forms $(\lambda-a)v = 0$, and $v(u+au') = 0$, respectively. Since (2) always passes through the centre of (1), $v = 0$ is also the locus of the centre of (1).]

9289. (R. Holmes, B.A.)—From the equations

$$C_{s-1}\cos ml_{s-1} - D_{s-1}\sin ml_{s-1} = C_s,$$

$$(C_{s-1}\sin ml_{s-1} + D_{s-1}\cos ml_{s-1})\,\omega_{s-1} = (C_s\sin ml_s + D_s\cos ml_s)\,\omega_s,$$

determine the ratio of C_s and D_s to D_1, being given that $C_1 = 0$; and hence prove the relation

$$\frac{D_1\omega_1\cos ml_1}{D_s\omega_s\cos ml_s} = 1 + \frac{C_s}{D_s}\tan ml_s.$$

Solution.

Since $$C_{s-1}\cos ml_{s-1} - D_{s-1}\sin ml_{s-1} = C_s,$$

and $$C_{s-1}\sin ml_{s-1} + D_{s-1}\cos ml_{s-1} = T/\omega_{s-1},$$

where $$T = (C_s\sin ml_s + D_s\cos ml_s)\,\omega_s = D_1\omega_1\cos ml_1;$$

we have, eliminating D_{s-1},

$$C_{s-1} = C_s \cos ml_{s-1} + (T/\omega_{s-1}) \sin ml_{s-1},$$

or $$C_s = -(T/\omega_{s-1}) \tan ml_{s-1} + \sec ml_{s-1}\, C_{s-1};$$

whence $$C_s = -T\{(1/\omega_{s-1}) \tan ml_{s-1} + (1/\omega_{s-2}) \tan ml_{s-2} \sec ml_{s-1} + (1/\omega_{s-3}) \tan ml_{s-3} \sec ml_{s-2} \sec ml_{s-1} + \dots + (1/\omega_1) \tan ml_1 \sec ml_2 \sec ml_3 \dots \sec ml_{s-1}\}$$
$$= -D_1\omega_1 \cos ml_1 \{\dots\dots\}.$$

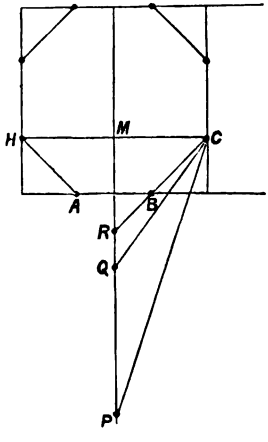
From the equation $C_s \sin ml_s + D_s \cos ml_s = (T/\omega_s)$

we have $D_s = (T/\omega_s) \sec ml_s - C_s \tan ml_s = (T/\omega_s) \sec ml_s + T \tan ml_s \{\dots\}$
$$= D_1\omega_1 \cos ml_1 [(1/\omega_s) \sec ml_s + \tan ml_s \{\dots\}].$$

Hence $$D_1\omega_1 \cos ml_1 = D_s\omega_s \cos ml_s + C_s\omega_s \sin ml_s.$$

9291. (J. O'Regan.)—Standing at an unknown distance from an octagonal house in the line perpendicular to one of its sides, I observed the angle subtended by the extreme visible corners; then, advancing 100 feet nearer in the same line, I found the same angle to be triple of what it was at the first station; and, advancing 20 feet still nearer, found I was in a line with two of the sides. Determine a side of the octagon.

Solution.



Let ABC ... H be the octagon; P, Q, R the three stations on the line bisecting AB at right angles, so that R, B, C and R, A, H are collinear. Also let 2α, 6α be the angles subtended by CH at P and Q respectively; CH = $2a$, M its middle point, RM = CM = a.

Then $$a(\cot\alpha - 1) = PR = 120,$$
$$a(\cot 3\alpha - 1) = QR = 20.$$

Hence $$\cot\alpha - 1 = 6(\cot 3\alpha - 1),$$

and it will be found this reduces to the equation

$$5\tan^3\alpha - 17\tan^2\alpha - 15\tan\alpha + 3 = 0,$$

which has three real roots: one positive and less than 1, another positive and greater than 1, and one negative,—of which the first is the one obviously required.

Approximating, by Horner's method, we get $\tan\alpha = \cdot 16917$, whence

$$2a = 240\tan\alpha/(1 - \tan\alpha) = 48\cdot 87.$$

If x be a side of the octagon, $x\{1 + 1/\sqrt{2} + 1/\sqrt{2}\} = 2a$, therefore

$$x = 2a(\sqrt{2} - 1) = 20\cdot 24 \text{ feet}.$$

9292. (J. L. Bury.)—A and B have a piece of work to do. B commences first and works for two-thirds of the time that it would take A to do the whole; then A sets to work and finishes it. Now, both working together could have done it two hours sooner, and B would then have done only half the quantity of work he left A to do. Show that A and B could each have done the work separately in 3 and 6 hours respectively.

Solution.

Let x, y be the times (in hours) it would take A and B respectively to do the work separately.

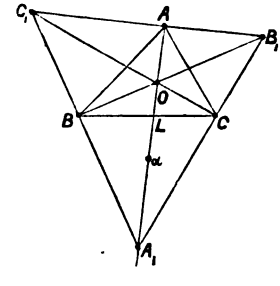
The x, y are determined by the relations

$$\{2x/3 + [(1-2x/3y)/1/x]\} - 2 = 1/(1/x + 1/y) = y/2\,(1-2x/3y),$$

whence $x = 3$, $y = 6$.

9295. (Emily Perrin, B.Sc.)—ABC is a triangle, O the centre of its inscribed circle, and A_1, B_1, C_1 the centres of the circles escribed to the sides BC, CA, AB respectively; L, M, N the points where these sides are cut by the bisectors of the angles A, B, C. Show that the orthocentres of the three triangles LB_1C_1, MC_1A_1, NA_1B_1 form a triangle similar and similarly situated to $A_1B_1C_1$, and having its orthocentre at O.

Solution.



Let α be the mid-point of OA_1, lying on the circum-circle. From the similarity of the triangles ABL, AαC, we have $AL \cdot A\alpha = AB \cdot AC$; and from the triangles AB_1C, ABC_1, we have

$$AB' \cdot AC' = AB \cdot AC.$$

Hence $AL \cdot A\alpha = AB_1 \cdot AC_1$;

and since AL is perpendicular to B_1C_1, therefore α is the orthocentre of the triangle LB_1C_1. Similarly, β, γ are the orthocentres of the triangles MC_1A_1, NA_1B_1; and

$$O\alpha : OA_1 = O\beta : OB_1 = O\gamma : OC_1,$$

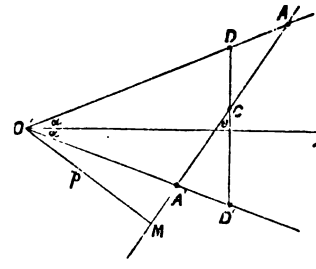
whence, &c.

9298. (The Editor.)—If a be the semi-vertical angle of a cone, p the perpendicular from the vertex on a plane which cuts a complete elliptic section from the cone, and θ the inclination of the axis of the cone to the cutting plane, show that the area (Σ) of the elliptic section is

$$\Sigma = \frac{\pi p^2 \sin^2 a \cos a}{\sin^{\frac{1}{2}}(\theta + a)\sin^{\frac{1}{2}}(\theta - a)} = \frac{2^{\frac{1}{2}}\pi p^2 \sin^2 a \cos a}{(\cos 2a - \cos 2\theta)^{\frac{1}{2}}}.$$

Solution.

Let the plane of the paper be that of a plane through the axis OX of the cone perpendicular to the sectional plane. These two planes intersect in a line AA′ (the major axis of the elliptic section) where OA, OA′ are the two generating lines of the cone lying in the former plane. Then θ is the angle between AA′, OX. Let C be the middle point of AA′, and DCD′ a line perpendicular to OX meeting OA, OA′ in D, D′.



The semi-axes of the elliptic section are AC ($= \frac{1}{2}$AA′), and $(CD \cdot CD')^{\frac{1}{2}}$.

Now $p = OM = OA \sin(\theta-\alpha) = AA' \sin(\theta+\alpha)\sin(\theta-\alpha) / \sin 2\alpha$,

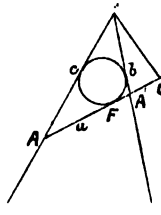
therefore $AA' = p \sin 2\alpha / \{\sin(\theta+\alpha)\sin(\theta-\alpha)\}$,

also $CD = AC \sin(\theta-\alpha)/\cos\alpha$; $CD' = A'C \sin(\theta+\alpha)/\cos\alpha$,

therefore $CD \cdot CD' = AC^2 \sin(\theta+\alpha)\sin(\theta-\alpha)/\cos^2\alpha$.

Hence we obtain $\Sigma = \pi \cdot AC \cdot (CD \cdot CD')^{\frac{1}{2}}$ = the given expressions.

[If AOA′ be the section of the cone by the plane of the paper, and a circle be inscribed touching AO, A′O, AA′, F will be the focus, and



$$\Sigma = \tfrac{1}{2}\pi a\{(s-b)(s-c)\}^{\frac{1}{2}} = \tfrac{1}{2}\pi \times \frac{bc\sin 2\alpha}{p} \times (bc)^{\frac{1}{2}}\sin\alpha;$$

also $$a = \frac{2\text{ area}}{p} = \frac{bc\sin 2\alpha}{p},$$

$$b = \frac{p}{\sin(\theta+\alpha)}, \quad c = \frac{p}{\sin(\theta-\alpha)};$$

thus $$\Sigma = \frac{\pi}{2p} \times \left(\frac{p}{\sin(\theta+\alpha)} \times \frac{p}{\sin(\theta-\alpha)}\right)^{\frac{1}{2}} \sin 2\alpha \sin\alpha = \frac{\pi p^2 \sin^2\alpha\cos\alpha}{\sin^{\frac{1}{2}}(\theta+\alpha)\sin^{\frac{1}{2}}(\theta-\alpha)}$$

The average of Σ for various positions of the cutting plane has been investigated in the Solution of Quest. 2600, on pp. 16—22 of Vol. XXI.]

9323. (E. Vigarié.)—Le point de Lemoine d'un triangle ABC, son point anticomplémentaire (point de Lemoine du triangle obtenu en menant par les sommets de ABC des parallèles aux côtés opposés), et les points réciproques (*isotomic points*) des points de Brocard sont les sommets d'un parallélogramme.

Solution.

Suppose three masses each $= a^2+b^2+c^2$ placed at the angular points of the triangle ABC. The centroid of this system is at G, the centroid of the triangle.

Now, the centroid of a^2 at A, b^2 at B, c^2 at C is at the Symmedian-point K.

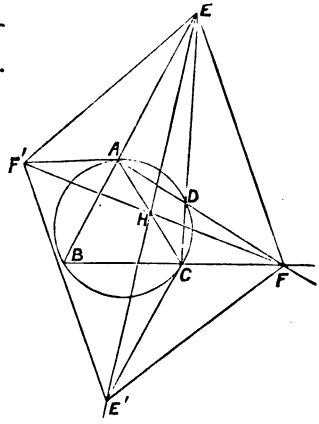
Since the centroid of $1/b^2$ at A, $1/c^2$ at B, $1/a^2$ at C, is the positive Brocard point Ω, the centroid of b^2 at A, c^2 at B, a^2 at C is at W, the iso-

tomic point to Ω. (CASEY'S *Sequel*, 4th edition, p. 169, Ex. 1, and p. 178, Ex. 7.)

Similarly, the centroid of c^2 at A, a^2 at B, b^2 at C is at W′, the isotomic point to Ω′. Hence G is the centroid of the triangle KWW′; and if KG be produced to K′, so that GK = 2KG, then KWK′W′ is a parallelogram. The point K′ is from this construction easily seen to be the Symmedian point of the triangle formed by drawing through the vertices of ABC parallels to the opposite sides.

9332. (G. S. CARR, M.A.)—If H be the mid-point of either diagonal of an in-quadrilateral of a circle, and if the sides produced meet in E and F, the two diagonals are proportional to the diagonals of the parallelogram on EH, HF.

Solution.



Let ABCD be the quadrilateral; H the mid-point of AC. Produce BA, CD to meet in E and AD, BC to meet in F. Upon FH produced through H take F′, so that FH = HF′; join EF, EF′. Then since AF′ is parallel to BC, the angle EAF′ = supplement of ABC = EDF. Also

$$EA : ED = AC : BD$$
$$= CF : DF = AF' : DF.$$

Hence the triangles EAF′, DF are similar; and therefore

EF′ : EF is also = AC : BD.

[The result of this neat solution gives at once a known value (not otherwise so easily obtainable) of the third diagonal in terms of interior dimensions. For, if K be the mid-point of BD, KH produced bisects EF, in L, say. Then, by the theorem, $\frac{AC}{BD} = \frac{HL}{EL} = \frac{EL}{KL}$;

therefore $$\frac{BD \cdot EL}{AC} - \frac{AC \cdot EL}{BD} = KL - HL = HK;$$

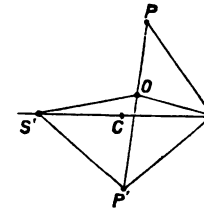
therefore $$EF \text{ or } 2EL = \frac{2HK \cdot AC \cdot BD}{AC^2 - BD^2}\Big].$$

9333. (G. H. BRYAN, B.A.)—The two diagonals and two opposite sides of a square are formed of four rigid bars freely jointed at their extremities. One diagonal is fixed and the other bars are moved about in their plane. Show that the middle point of the other diagonal describes a lemniscate.

Solution.

Let SS′ be the fixed diagonal, C its mid-point; SP, S′P′ the opposite sides turning about S, S′ respectively; and O the mid-point of the diagonal PP′. Then in any position the sides of the triangle SPP′ are equal to those of the triangle P′S′S, therefore SPS′P′ are concyclic, and PP′ is the chord of the circum-circle of SPS′, which is the symmedian line at P. The mid-point O is the "permanent centre of similitude" of figures upon SP, PS′, and

$$SO \cdot S'O = OP^2 = CS^2 = \text{constant}.$$



This is a particular case of the general theorem that in any motion produced by a "crossed rhomboid" (or "contra-parallelogram") any point in the moving plane describes the inverse of a conic.

9335. (The Editor.)—If from a point Q on a circle AQX, a perpendicular QR be drawn to the diameter AX, and AP be drawn to touch at P a circle with QR as diameter and O as centre, and PG be a perpendicular on QR, prove that (RG : GQ) = 4 (AR : RX).

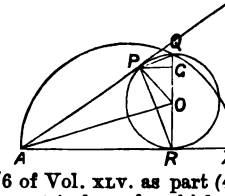
Solution.

By similar triangles, we have

$$PR^2 : 4AR^2 = PQ^2 : QR^2,$$

or $$QR \cdot RG : 4AR^2 = GQ \cdot QR : AR \cdot RX,$$

or $$RG : 4AR = GQ : RX.$$



[This theorem is proved *analytically* on p. 76 of Vol. XLV. as part (4) of Quest. 7896. The first line of Mr. Davis's *geometrical* proof—which is not quite obvious—may be seen from the fact that, since a circle will go round APOR, ∠OAR = OPR = ORP, hence PQ : PR = OR : AR = QR : 2AR.]

9342. (Asparagus.)—The fourth common tangent to an ellipse and its circle of curvature at a point P touches the ellipse in Q; prove that, the coordinates of P being $(a\cos\theta, b\sin\theta)$, the equation of PQ is

$$x/a\cos\theta\,(1-2e^2\cos^2\theta+e^2\cos^4\theta)-y/b\sin\theta\,(1-e^2\cos^4\theta) + 1-2\cos^2\theta+e^2\cos^4\theta = 0.$$

Solution.

Let the tangent to the ellipse at P, Q intersect in T (h, k). Then T lies on the confocal hyperbola through P (Taylor's *Geometry of Conics*, Chapter VIII., Ex. 644), and $h^2\sec^2\theta - k^2\operatorname{cosec}^2\theta = c^2$. Combining this with $h/a\cos\theta + k/b\sin\theta = 1$, we get

$$h = -a\cos\theta\left\{1-2e^2\cos^2\theta+e^2\cos^4\theta\right\}/\mathrm{U}, \quad k = b\sin\theta\left\{1-e^2\cos^4\theta\right\}/\mathrm{U},$$

where $$\mathrm{U} = 1-2\cos^2\theta+e^2\cos^4\theta.$$

Substituting these values of h, k in the equation to the polar of T, $xh/a^2+yk/b^2 = 1$, we get the required result.

APPENDIX III.

SOLUTIONS OF SOME OLD QUESTIONS.

By W. J. Curran Sharp, M.A.

2926 & 6306. (Professor Sylvester, F.R.S.)—(2926) Let f, g, h, k be any four given unequal quantities, and m, n, p, q four others (to be found) such that the arc of the curve

$$y = f\log(x^2-m^2) + g\log(x^2-n^2) + h\log(x^2-p^2) + k\log(x^2-q^2)$$

is equal to $\quad x + f\log\frac{x+m}{x-m} + g\log\frac{x+n}{x-n} + h\log\frac{x+p}{x-p} + k\log\frac{x+q}{x-q}$.

Prove that, in general, there are nine systems of values of m, n, p, q which will satisfy the above condition.

(6306.) Given the system of equations

$$x_1(x_1+x_2)(x_1+x_3)\dots(x_1+x_n) = C_1(x_1-x_2)(x_1-x_3)\dots(x_1-x_n),$$
$$x_2(x_2+x_1)(x_2+x_3)\dots(x_2+x_n) = C_2(x_2-x_1)(x_2-x_3)\dots(x_2-x_n),$$
$$\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots$$
$$x_n(x_n+x_1)(x_n+x_2)\dots(x_n+x_{n-1}) = C_n(x_n-x_1)(x_n-x_2)\dots(x_n-x_{n-1});$$

prove that the number of solutions cannot exceed $1^2.3^2.5\dots(n-1)^2$ or $1^2.3^2.5^2\dots(n-2)^2 n$, according as n is even or odd.

Solution.

If y and s have the given values, we have

$$\frac{dy}{dx} = 2\left\{\frac{fx}{x^2-m^2} + \frac{gx}{x^2-n^2} + \frac{hx}{x^2-p^2} + \frac{kx}{x^2-q^2}\right\},$$

$$\frac{ds}{dx} = 1 - 2\left\{\frac{fm}{x^2-m^2} + \frac{gn}{x^2-n^2} + \frac{hp}{x^2-p^2} + \frac{kq}{x^2-q^2}\right\},$$

therefore
$$1 \equiv \left(\frac{ds}{dx}\right)^2 - \left(\frac{dy}{dx}\right)^2 \equiv \left(\frac{ds}{dx}+\frac{dy}{dx}\right)\left(\frac{ds}{dx}-\frac{dy}{dx}\right)$$
$$= \left[1 + 2\left\{\frac{f}{x+m} + \frac{g}{x+n} + \frac{h}{x+p} + \frac{k}{x+q}\right\}\right]$$
$$\times \left[1 - 2\left\{\frac{f}{x-m} + \frac{g}{x-n} + \frac{h}{x-p} + \frac{k}{x-q}\right\}\right]$$

$$\equiv 1-4\left\{\frac{fm}{x^2-m^2}+\frac{gn}{x^2-n^2}+\frac{hp}{x^2-p^2}+\frac{kq}{x^2-q^2}\right\}$$

$$-4\left\{\frac{f}{x+m}+\frac{g}{x+n}+\frac{h}{x+p}+\frac{k}{x+q}\right\}\left\{\frac{f}{x-m}+\frac{g}{x-n}+\frac{h}{x-p}+\frac{k}{x-q}\right\};$$

therefore $\frac{fm}{x^2-m^2}+\frac{gn}{x^2-n^2}+\frac{hp}{x^2-p^2}+\frac{kq}{x^2-q^2}+\left\{\frac{f}{x+m}+\frac{g}{x+n}+\frac{h}{x+p}+\frac{k}{x+q}\right\}$

$$\times\left\{\frac{f}{x-m}+\frac{g}{x-n}+\frac{h}{x-p}+\frac{k}{x-q}\right\}\equiv 0;$$

or $fm\,(x^2-n^2)\,(x^2-p^2)\,(x^2-q^2)$ + &c.

$$+\left\{f(x+n)\,(x+p)\,(x+q)+\&c.\right\}\left\{f(x-n)\,(x-p)\,(x-q)+\&c.\right\}\equiv 0,$$

an equation which holds for all values of x, but is nugatory if any two of m, n, p, and q be equal.

If $x = m$, this gives

$$1+2\left\{\frac{f}{m+m}+\frac{g}{m+n}+\frac{h}{m+p}+\frac{k}{m+q}\right\}=0.$$

Similarly, $1+2\left\{\frac{f}{n+m}+\frac{g}{n+n}+\frac{h}{n+p}+\frac{k}{n+q}\right\}=0$, &c.

So that m, n, p, and q are the roots of the equation

$$1+2\left(\frac{f}{y+m}+\frac{g}{y+n}+\frac{h}{y+p}+\frac{k}{y+q}\right)=0,$$

or $(y+m)\,(y+n)\,(y+p)\,(y+q)+2f\,(y+n)\,(y+p)\,(y+q)$

$$+2g\,(y+p)\,(y+q)\,(y+m)+\&c.=0,$$

therefore $m+n+p+q+2f+2g+2h+2k=-m-n-p-q$ .....(1);

$$mn+np+pm+mq+nq+pq+2f(n+p+q)+2g\,(p+q+m)$$

$$+2h(q+m+n)+2k(m+n+p)=mn+np+pm+mq+nq+pq\ (2);$$

$$mnp+npq+pqm+qmn+2f\,(np+pq+qn)+2g\,(pq+qm+mp)$$

$$+2h\,(qm+mn+nq)+2k\,(mn+np+pm)=-mnp-npq-pqm-qmn\ldots\ (3);$$

and $mnpq+2fnpq+2gpqm+2hqmn+2kmnp=mnpq$ ..... (4),

of which (1) and (2) are linear equations in m, n, p, and q, and (3) and (4) are cubic equations, so that there are, in general, nine solutions.

If in the identity

$$(y+m)(y+n)(y+p)(y+q)+2f\,(y+n)\,(y+p)(y+q)+2g\,(y+p)(y+q)(y+m)$$

$$+2h\,(y+q)\,(y+m)(y+n)+2k\,(y+m)(y+n)\,(y+p)$$

$$\equiv(y-m)\,(y-n)\,(y-p)\,(y-q),$$

y be put $= -m$, $-n$, $-p$, $-q$ successively, the results will be

$$m\,(m+n)\,(m+p)\,(m+q)=-f\,(m-n)\,(m-p)\,(m-q),$$

$$n\,(n+p)\,(n+q)\,(n+m)=-g\,(n-p)\,(n-q)\,(n-m),$$

&c. &c.,

results which may also easily be derived from the previous system of equations to which they are equivalent.

These equations show that this question is a particular case of Question 6306, while they explain the origin of the equations of that question.

For, if $y = a_1 \log(x^2 - x_1^2) + a_2 \log(x^2 - x_2^2) + \ldots + a_n \log(x^2 - x_n^2)$,

and $$s = x + a_1 \log \frac{x+x_1}{x-x_1} + a_2 \log \frac{x+x_2}{x-x_2} + \ldots + a_n \log \frac{x+x_n}{x-x_n},$$

therefore
$$1 \equiv \left(\frac{ds}{dx} + \frac{dy}{dx}\right)\left(\frac{ds}{dx} - \frac{dy}{dx}\right)$$
$$\equiv \left\{1 + 2\sum_{r=1}^{r=n} \frac{a_r}{x+x_r}\right\}\left\{1 - 2\sum_{r=1}^{r=n} \frac{a_r}{x-x_r}\right\}$$
$$\equiv 1 - 4\sum_{r=1}^{r=n} \frac{a_r x_r}{x^2 - x_r^2} - 4\sum_{r=1}^{r=n} \frac{a_r}{x+x_r} \cdot \sum_{r=1}^{r=n} \frac{a_r}{x-x_r};$$

therefore
$$\sum_{r=1}^{r=n} \frac{a_r x_r}{x^2 - x_r^2} + \sum_{r=1}^{r=n} \frac{a_r}{x+x_r} \cdot \sum_{r=1}^{r=n} \frac{a_r}{x-x_r} \equiv 0,$$

or $$a_1 x_1 (x^2 - x_2^2) \ldots (x^2 - x_n^2) + a_2 x_2 (x^2 - x_3^2) \ldots (x^2 - x_n^2)(x^2 - x_1^2) + \ldots$$
$$+ \{a_1 (x+x_2)(x+x_3) \ldots (x+x_n) + a_2 (x+x_3) \ldots (x+x_n)(x+x_1) + \ldots\}$$
$$\times \{a_1 (x-x_2)(x-x_3) \ldots (x-x_n) + a_2 (x-x_3) \ldots (x-x_n)(x-x_1) + \ldots\} \equiv 0;$$

or, putting $x = x_1$, &c.,
$$x_1 (x_1+x_2)(x_1+x_3) \ldots (x_1+x_n)$$
$$+ \{a_1 (x_1+x_2)(x_1+x_3) \ldots (x_1+x_n) + a_2 (x_1+x_3) \ldots (x_1+x_n)(x_1+x_1) + \&c.\} = 0,$$
&c. &c.;

or $$1 + 2\left\{\frac{a_1}{x_1+x_1} + \frac{a_2}{x_1+x_2} + \ldots\right\} = 0, \quad 1 + 2\left\{\frac{a_1}{x_2+x_1} + \frac{a_2}{x_2+x_2} + \ldots\right\} = 0,$$
&c. &c.

So that x_1, x_2, &c. are the roots of
$$1 + 2\left\{\frac{a_1}{y+x_1} + \frac{a_2}{y+x_2} + \ldots\right\} = 0;$$

or, writing $a_1 = -c_1$, $a_2 = -c_2$, &c., x_1, x_2, &c. are the roots of
$$1 - 2\left\{\frac{c_1}{y+x_1} + \frac{c_2}{y+x_2} + \ldots\right\} = 0;$$

and
$$(y+x_1)(y+x_2) \ldots (y+x_n)$$
$$- 2\{c_1 (y+x_2)(y+x_3) \ldots (y+x_n) + c_2 (y+x_3) \ldots (y+x_n)(y+x_1) + \&c.\}$$
$$\equiv (y-x_1)(y-x_2) \ldots (y-x_n).$$

If y be put equal to $-x_1$, $-x_2$, &c., in succession, the equations of (6306) are obtained, while, by equating coefficients, a series of equations of order 1, 1, 3, 3, 5, 5, &c. in $x_1, x_2 \ldots x_n$ is obtained, and so the number of solutions cannot exceed $1^2 . 3^2 \ldots (x-1)^2$ when n is even, or $1^2 . 3^2 \ldots (n-2)^2 . n$ when n is odd.

5806. (W. J. C. SHARP, M.A.)—Find (1) the law of distribution of mass such that the catenary (under the action of gravity) may be a given curve, for example, the portion of an ellipse cut off by its horizontal major axis; and (2) supposing the mass supplied by a fringe of uniform separate threads, find the locus of the ends both generally and in the particular case.

Solution.

If μds be the mass of the element ds of the string, and if the axes of x and y be taken horizontal and vertical (positive direction upwards), then (MINCHIN'S *Statics*, p. 337),

$$\frac{d}{ds}\left(\mathrm{T}\frac{dx}{ds}\right) = 0 \text{ and } \frac{d}{ds}\left(\mathrm{T}\frac{dy}{ds}\right) = \mu g,$$

therefore $\frac{dx}{ds} = \frac{c}{\mathrm{T}}$, where c is an arbitrary constant, and therefore

$$c\frac{dx}{ds}\cdot\frac{d^2y}{dx^2} = \mu g \text{ or } \mu \propto \frac{dx}{ds}\cdot\frac{d^2y}{dx^2},$$

and $\frac{dx}{ds}$ and $\frac{d^2y}{ds^2}$ is known from the equation to the curve.

Now, if $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$\frac{dx}{ds} = \frac{a(a^2-x)^{\frac{1}{2}}}{(a^4 - a^2x^2 + b^2x^2)^{\frac{1}{2}}} \text{ and } \frac{d^2y}{dx^2} = -\frac{a^2b}{(a^2-x^2)^{\frac{3}{2}}},$$

and therefore $$\mu \propto \frac{1}{(a^2-x^2)(a^4 - a^2x^2 + b^2x^2)^{\frac{1}{2}}}.$$

Now, if (x, y') be the point on the end of the fringe corresponding to (x, y)

$$y' = y + c\mu = y + \lambda\frac{dx}{ds}\cdot\frac{d^2y}{dx^2},$$

and the equation to the locus of (x, y') is formed by eliminating y by help of the given equation to the catenary. In the particular case

$$y' = -\frac{b}{a}(a^2-x^2)^{\frac{1}{2}} + \frac{\lambda}{(a^2-x^2)(a^4 - a^2x^2 + b^2x^2)^{\frac{1}{2}}}.$$

6271. (W. J. C. SHARP, M.A.)—Of all the lines whose satellites, with respect to a nodal cubic, pass through the node, prove that each passes through one of four fixed points on the harmonic polar of the real inflexion.

Solution.

I have shown (*Quarterly Journal*, Vol. XVI., p. 186), that the equation to any proper cubic may be expressed in the form

$$ax^3 + 3c_1xz^2 + cz^3 + 3b_3y^2z = 0,$$

where $x = 0$, $z = 0$ is an inflexion, and $y = 0$ its harmonic polar; that, if

this cubic be nodal, $ac^2 + 4c_1^3 = 0$, the node being determined by the equations $x + \frac{c}{2c_1} z = 0$, $y = 0$; and that, if $\lambda x + \mu y + \nu z = 0$ be the satellite of $\alpha x + \beta y + \gamma z = 0$ (p. 299),

$$\begin{aligned}\lambda : \mu : \nu = &-4ab_3^2c\alpha^4 + 12ab_3^2c_1\alpha^3\gamma + 12ab_3c_1^2\alpha^2\beta^2 \\ &+ 12a^2b_3c\alpha\beta^2\gamma + 4a^2b_3^2\alpha\gamma^3 - 12a^2b_3c_1\beta^2\gamma^2 \\ : &\ 8ab_3^2c\alpha^3\beta - 24ab_3^2c_1\alpha^2\beta\gamma - 8a^2b_3^2\beta\gamma^3 \\ : &\ 9b_3^2c_1^2\alpha^4 + 8ab_3^2c\alpha^3\gamma + 6ab_3cc_1\alpha^2\beta^2 \\ &- 24ab_3c_1^2\alpha\beta^2\gamma - 3\,(a^2c^2 + 4ac_1^3)\,\beta^4 - 6a^2b_3c\beta^2\gamma^2 + a^2b_3^2\gamma^4.\end{aligned}$$

Hence, if $\lambda x + \mu y + \nu z = 0$ pass through the node, $\lambda c = 2\nu c_1$. Then, remembering that $ac^2 + 4c_1^3 = 0$,

$$c_1^3\alpha^4 + 2acc_1\alpha^2\gamma - 2a^2c\alpha\gamma^3 + a^2c_1\gamma^4 = 0,$$

a quartic equation to determine $\alpha : \gamma$; and therefore, &c.

6371. (W. J. C. Sharp, M.A.)—If a uniform wire, of such a nature that the resistance to bending varies as the curvature, be suspended, without torsion or extension of the mean fibre, from two points; show that it will hang in a catenary identical with that in which a uniform string of the same length would hang, but that the tension along the wire will differ by a constant quantity from that along the string.

Solution.

If F denote the normal force of resistance to bending at the point (x, y, z), this will act along the principal normal, since the bending takes place in the osculating plane; and therefore the components of this force are $\mathrm{F}\rho\frac{d^2x}{ds^2}$, $\mathrm{F}\rho\frac{d^2y}{ds^2}$, $\mathrm{F}\rho\frac{d^2z}{ds^2}$, parallel to the axes.

Hence, if the external impressed forces on the element μ at (x, y, z) be X, Y, Z the ordinary equations of equilibrium, viz.,

$$\frac{d}{ds}\left(\mathrm{T}\frac{dx}{ds}\right) + \mu\mathrm{X} = 0, \quad \frac{d}{ds}\left(\mathrm{T}\frac{dy}{ds}\right) + \mu\mathrm{Y} = 0, \quad \frac{d}{ds}\left(\mathrm{T}\frac{dz}{ds}\right) + \mu\mathrm{Z} = 0,$$

are replaced by

$$\frac{d}{ds}\left(\mathrm{T}\frac{dx}{ds}\right) + \mathrm{F}\frac{d^2x}{ds^2} + \mu\mathrm{X} = 0, \quad \frac{d}{ds}\left(\mathrm{T}\frac{dy}{ds}\right) + \mathrm{F}\rho\frac{d^2y}{ds^2} + \mu\mathrm{Y} = 0,$$

$$\frac{d}{ds}\left(\mathrm{T}\frac{dz}{ds}\right) + \mathrm{F}\rho\frac{d^2z}{ds^2} + \mu\mathrm{Z} = 0,$$

and, if $\mathrm{F}\rho = f$ = constant, these may be written

$$\frac{d}{ds}\left\{(\mathrm{T}+f)\frac{dx}{ds}\right\} + \mu\mathrm{X} = 0, \quad \frac{d}{ds}\left\{(\mathrm{T}+f)\frac{dy}{ds}\right\} + \mu\mathrm{Y} = 0, \quad \frac{d}{ds}\left\{(\mathrm{T}+f)\frac{dz}{ds}\right\} + \mu\mathrm{Z} = 0,$$

which only differ from those for a flexible string by the substitution of $\mathrm{T}+f$ for T, and therefore, etc.

8353. (Professor Booth, M.A.)—Eliminate (1) x, y, z from the equations $ax^2+2hxy+by^2=0$, $by^2+2fyz+cz^2=0$, $cz^2+2gzx+ax^2=0$, and give (2) a geometrical interpretation of the result.

Solution.

If the given equations be written

$$-2f=\frac{by}{z}+\frac{cz}{y},\quad -2g=\frac{cz}{x}+\frac{ax}{z},\quad -2h=\frac{ax}{y}+\frac{by}{x};$$

therefore $$4fg=\frac{bcy}{x}+\frac{acx}{y}+\frac{c^2z^2}{xy}+\frac{abxy}{z^2}=-2ch+\frac{c^2z^2}{xy}+\frac{abxy}{z^2};$$

therefore $$-4h(2fg+ch)=\left(\frac{c^2z^2}{xy}+\frac{abxy}{z^2}\right)\left(\frac{ax}{y}+\frac{by}{x}\right)$$

$$=a\left(\frac{c^2z^2}{y^2}+\frac{b^2y^2}{z^2}\right)+b\left(\frac{a^2x^2}{z^2}+\frac{c^2z^2}{x^2}\right)$$

$$=a(4f^2-2bc)+b(4g^2-2ac),$$

therefore $$abc-2fgh-af^2-bg^2-ch^2=0,$$

a result given in the solution of Question 6813 (Vol. xxxvii., p. 21), which is the discriminant of

$$ax^2+by^2+cz^2-2fy^2-2gzx-2hxy=0,$$

and hence follows the converse of Question 7585 (Vol. xlii., p. 124), viz.

If two sets of concurrent lines be drawn from the vertices of a triangle to meet the opposite sides, the harmonic conjugates of the feet of these lines with respect to the vertices which are on the same side of the triangle, lie on two straight lines.

8567. (Rev. T. P. Kirkman, M.A., F.R.S.)—If, in any 4-edron $abcd$, ca be the 2-edral angle about ac, prove that

$$[\cos ca+\cos(cb-cd)]\,[(bd)^2-(cb-cd)^2]$$
$$=[\cos ca+\cos(ad-ab)]\,[bd^2-(ad-ab)^2].$$

Solution.

By Spherical Trigonometry, we have

$$\cos ca=-\cos cb\cos cd+\sin cb\sin cd\cos dcb,$$

therefore $$\frac{\cos ca+\cos cb\cos cd}{\sin cb\sin cd}=\cos dcb,$$

therefore $$\frac{\cos ca + \cos(cb-cd)}{\sin cb \sin cd} = 1 + \cos dcb$$

$$= \frac{(dc+bc)^2-(bd)^2}{2cb \,.\, cd} \text{ (by Plane Trigonometry),}$$

therefore $$\{\cos ca + \cos(cb-cd)\}\ \{(bd)^2-(cb-cd)^2\}$$

$$= -\,8\,\frac{\sin cb \sin cd}{cb \,.\, cd}\,(\Delta bcd)^2.$$

Similarly $$\{\cos ca + \cos(ab-ad)\}\ \{(bd)^2-(ab-ad)^2\}$$

$$= -\,8\,\frac{\sin ab \sin ad}{ab \,.\, ad}\,(\Delta abd)^2,$$

and the perpendicular from d on abc

$$= (\text{perpendicular from } d \text{ on } ab)\sin ab = \frac{\Delta abd}{2ab}\sin ab,$$

or $$= (\text{perpendicular from } d \text{ on } bc)\sin bc = \frac{\Delta bcd}{2bc}\sin bc,$$

therefore $$\frac{\Delta abd}{2ab}\sin ab = \frac{\Delta bcd}{2bc}\sin bc.$$

Similarly $$\frac{\Delta abd}{2ad}\sin ad = \frac{\Delta bcd}{2cd}\sin cd,$$

and therefore

$$\{\cos ca + \cos(cb-cd)\}\ \{(bd)^2-(cb-cd)^2\}$$

$$= \{\cos ca + \cos(ad-ab)\}\ \{(bd)^2-(ad-ab)^2\}.$$

From the above it appears that

$$\frac{\sin ab}{ab} : \frac{\sin ad}{ad} = \frac{\sin cb}{cb} : \frac{\sin cd}{cd}.$$

8586. (Professor Sylvester, F.R.S.)—Assuming that the pressure of fluids is transmitted equally in all directions, prove the following theorem in Statics:—Any system of forces respectively perpendicular and proportional to the faces of a polyhedron is in equilibrium.

Solution.

If we imagine a weightless fluid shut up in a closed vessel and pressed upon by one or more pistons in the boundary of the vessel, the pressure at every point of the fluid will be the same, and the fluid will be in equilibrium. Also the pressure on any plane surface will be proportional to its area, and will act perpendicularly to it through its centroid. If, then, a polyhedron of the fluid be solidified, it will be in equilibrium under the action of pressures proportional to the faces and acting through their centroids perpendicularly to them.

7488. (Professor HUDSON, M.A.)—If O be the circumcentre of ABC, and forces act along OA, OB, OC proportional to BC, CA, AB; prove that their resultant passes through the in-centre.

7533. (J. J. WALKER, M.A., F.R.S.)—Prove that the common centroid of the surface-mass of the four faces of a tetrahedron is the centre of the sphere inscribed in that determined by the four centroids of the faces; and hence prove the obvious analogue in tri-dimensional space of Professor HUDSON's question 7488—which is true in any position of the point O, for forces proportional to OA sin A, OB sin B, OC sin C.

8242. (Professor SYLVESTER, F.R.S.)—If by a Simplicissimum of the nth order be understood a figure in space of n dimensions formed by the indefinite protraction of the series of which a linear segment, a triangle, and a pyramid are the three first terms, prove that (1) when each edge is unity, the squared content is $\frac{(n+1)}{2^n(1\,.\,2\,.\,3\,\ldots\,n)^2}$; and hence (2) deduce that the Cayleyan Persymmetrical Invertebrate Determinant of the squared edges by which such squared content is imaged must be diminished in the ratio of negative unity to $(-2)^n(1\,.\,2\,.\,3\,\ldots\,n)^2$, in order that it may represent its absolute value. *Ex. gr.*, the determinant

$$\begin{vmatrix} . & (ab)^2 & (ac)^2 & 1 \\ (ba)^2 & . & (bc)^2 & 1 \\ (ca)^2 & (cb)^2 & . & 1 \\ 1 & 1 & 1 & . \end{vmatrix}$$

images the squared contents of a triangle whose edges are (ab), (ac), (bc), for the triangle vanishes when this determinant vanishes, but the actual value of the squared content is this determinant diminished in the ratio of negative unity to $2^2(1\,.\,2)^2$, *i.e.*, multiplied by $-\frac{1}{16} = (2)$.

8216. (Professor SYLVESTER, F.R.S.)—Let x, y, z, t be the distances of any point P, and x', y', z', t' of P′, from the faces of a fundamental pyramid divided respectively by the distances of the vertices a, b, c, d of the pyramid from those faces; prove that $PP'^2 = \Sigma(ab)^2(x-x')(y'-y)$. [Of course a precisely similar formula applies to distances on a plane or in space of any number of dimensions.]

8042. (Professor SYLVESTER, F.R.S.)—Let A, B, C, D be the perpendiculars upon a plane from the points a, b, c, d, the angles of a pyramid whose volume is P. Required (1) to prove that

$$\Sigma(ab)^4(C-D)^2 - 2\Sigma(ab)^2(ac)^2(D-B)(D-C)$$
$$+ 2\Sigma(ab)^2(cd)^2(A-C)(A-D) + (B-C)(B-D) = -144P^2.$$

Show also (2) how to find the Constant Homogeneous Quadratic Function of the five perpendiculars from five points in space of four dimensions upon any hyper-plane drawn thereon.

8078. (Professor SYLVESTER, F.R.S.)—If in a system of quadruplanar coordinates, for which $x+y+z+t$ expresses the plane at infinity, $abcd$ is the pyramid of reference; show that (1) $\Sigma(ab)^2xy$ is the sphere which circumscribes it; and hence, (2) if ξ, η, θ, ζ are the perpendicular

distances of a, b, c, d from any variable plane, the following determinant is a constant, and find its value :—

$$\begin{vmatrix} . & \xi & \eta & \theta & \zeta & . \\ \xi & . & (ab)^2 & (ac)^2 & (ad)^2 & 1 \\ \eta & (ba)^2 & . & (bc)^2 & (bd)^2 & 1 \\ \theta & (ca)^2 & (cb)^2 & . & (cd)^2 & 1 \\ \zeta & (da)^2 & (db)^2 & (dc)^2 & . & 1 \\ . & 1 & 1 & 1 & 1 & . \end{vmatrix}$$

8976. (W. J. C. Sharp, M.A.)—Show that, if (1 . 2), (2 . 3), &c. denote the edges of a simplicissimum in space of n dimensions (see Quest. 8242), the squared radius of the circumscribed hyper-sphere is

$$-\tfrac{1}{2}\begin{vmatrix} 0, & (1\,.\,2)^2, & (1\,.\,3)^2\ldots \\ (1\,.\,2)^2, & 0, & (2\,.\,3)^2\ldots \\ \ldots & \ldots & \ldots \\ (1.n+1)^2, & (2.n+1)^2, & (3.n+1)^2\ldots \end{vmatrix} + \begin{vmatrix} 0, & 1, & 1, & 1\ldots \\ 1, & 0, & (1\,.\,2)^2, & (1\,.\,3)^2\ldots \\ 1, & (1\,.\,2)^2, & 0, & (2\,.\,3)^2\ldots \\ & \ldots & \ldots & \ldots \\ 1, & (1.n+1)^2, & (2.n+1)^2, & (3.n+1)^2\ldots \end{vmatrix}$$

which reduces to $\dfrac{n}{2\,(n+1)}$ when each of the edges is unity.

Also, show that, if d be the distance of the mean centre of equal masses at the vertices of the simplicissimum from the centre of the hyper-sphere, the same quantity equals $d^2 + \dfrac{1}{(n+1)^2}\{(1\,.\,2)^2 + (2\,.\,3)^2 + (3\,.\,1)^2 + \&c.\}$.

9098. (W. J. C. Sharp, M.A.)—If a tetrahedron be such that the sums of the squares of its opposite edges are all equal, show that the nine-point circles of the triangles, which are its faces, are all sections of the same sphere. Also show, in space of n dimensions, that if (r, s) denote the line joining the r^{th} and s^{th} vertices of a simplicissimum, and if A_1, $A_2 \ldots A_{n+1}$ be $(n+1)$ magnitudes, such that $(r, s)^2 = A_r + A_s$ for all the values of r and s, the nine-point circles of all the triangles, which can be formed so as to have their vertices at three of those of the simplicissimum (see Quest. 8242), lie upon the same hyper-sphere.

4208. (H. S. Monck, M.A.)—Three lines AL, BM, CN are drawn from the vertices of a triangle ABC through a point O to the opposite sides; prove that $2\Delta ABC : \Delta LMN = AO\,.\,BO\,.\,CO : LO\,.\,MO\,.\,NO$.

7335. (W. J. C. Sharp, M.A.)—If O be the circum-centre of the triangle ABC, and AO, BO, CO be produced to meet the opposite sides in D, E, F, the circle in D′, E′, F′ respectively; prove that

$$\frac{DD'}{AD} + \frac{EE'}{BE} + \frac{FF'}{CF} = 1, \quad BD\,.\,DC + OD^2 = AE\,.\,EC + OE^2 = AF\,.\,FB + OF^2.$$

7660. (R. Knowles, B.A., L.C.P.)—From the angular points of a triangle ABC, lines are drawn through the circum-centre to meet the opposite sides in D, E, F, respectively; prove that

$$\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF} = \frac{2}{R}.$$

7954. (W. J. C. SHARP, M.A.)—In a triangle ABC, if p_1 be the perpendicular from A upon BC, r the in-radius and r_1 the ex-radius to BC, show that (1) $\frac{1}{r} - \frac{1}{r_1} = \frac{2}{p_1}$; (2) the same equation holds if p_1 be the perpendicular from the vertex A of a tetrahedron upon the opposite face, and r the radius of the inscribed sphere, and r_1 that of the sphere touching BCD and the other faces produced.

7980. (W. J. C. SHARP, M.A.)—If AA′, BB′, &c. be concurrent lines drawn from the vertices of any tetrahedron ABCD to meet the opposite faces in A′, B′, &c., show that

triangle A′BC : triangle A′DB = triangle B′AC : triangle B′DA,

triangle A′BD : triangle A′CD = triangle D′BA : triangle D′AC, &c.

8927. (S. TEBAY, B.A.)—If Δ_1, Δ_2, Δ_3, Δ_4 be the areas of the faces of a tetrahedron, R the radius of the circumscribing sphere, and R_1, R_2, R_3, R_4 the radii of spheres passing through the centre of R and the angles of Δ_1, Δ_2, Δ_3, Δ_4; the volume

$$V = \tfrac{1}{6}R^2\left(\frac{\Delta_1}{R_1} + \frac{\Delta_2}{R_2} + \frac{\Delta_3}{R_3} + \frac{\Delta_4}{R_4}\right).$$

Solution.

(7488 & 7533). MR. WALKER's Question (7533) maybe further generalised into the following :—" Prove that the centroid of the faces of a simplicissimum is the centre of the spheric (hyper-sphere) inscribed in the derived simplicissimum, whose vertices are at the centroids of the faces; and hence prove the analogue in n-dimensional space of Professor HUDSON's Question 7488, which is true for any position of the point O, if the forces be proportional to $OA \cdot V_1$, $OB \cdot V_2$, &c., where V_1, V_2, &c. are the contents of the faces of the simplicissimum." A Solution of this and the other Questions given above, will be found in a paper by me, "On the Properties of Simplicissima, &c.," printed in the *Proceedings of the London Mathematical Society*, Vol. XVIII., p. 357.

For the derived simplicissimum is evidently similar to the original figure, and hence the centroid of all the faces, except the one upon $\lambda = 0$, is the foot of the line equidistant from all the faces of the derived figure which meet upon $\lambda = 0$, and so for the others, and the centroid which lies upon each of these lines is the centre of the spheric (hyper-sphere) inscribed in the derived simplicissimum.

Again, if from any point O lines be drawn to the vertices of any simplicissimum (A, B, &c.), and forces proportional to $OA \cdot V_1$, $OB \cdot V_2$, &c., act along these, the resultant will act along OG_1, where G_1 is the centroid of V_1 at A, V_2 at B, &c., which is by the above the centre of the spheric inscribed in the simplicissimum, and its magnitude will be proportional to

$$(V_1 + V_2 + \ldots V_n)\, OG_1.$$

(8242). For a solution of this germ Question, see p. 327 of the above-cited paper.

(8216). For a solution of this Question, see p. 343 of the above-cited paper.

(8042 & 8078). For a solution of these Questions, see pp. 352, 353 of the above-cited paper.

(8976). For a solution of this Question, see pp. 327 and 329 of the above-cited paper.

(9098). For a solution of this Question, see pp. 357, 358 of the above-cited paper, and p. 173 of this Volume.

(4208). For generalisations and solutions of this Question, see p. 339 of the above-cited paper.

(7335 & 7660). For a solution of these Questions, see p. 346 of the above-cited paper.

(7954). For a solution of this Question, see p. 355 of the above-cited paper.

(7980). For a solution of this Question, see p. 338 of the above-cited paper.

(8927). For a solution of this Question, see pp. 331, 332, of the above-cited paper.

6386. (W. S. McCay, M.A.)—If two spherics cut orthogonally, prove that (1) an indefinite number of pairs of points can be found on their common diameter, such that either point has the same polar to one of the spherics as the other has to the other, and (2) the distance between such a pair of points subtends a right angle at any intersection of the spherics.

Solution.

Let $$x^2+y^2+z^2 \ldots\ldots\ldots\ldots -r^2 = 0 \equiv C,$$
$$x^2+y^2+z^2+\ldots -2hx+r^2 = 0 \equiv C',$$

be the spherics. The linear polar of $(x_1, 0, 0 \ldots)$ with respect to C is $xx_1-r^2 = 0$, and that of $(x_2, 0, 0 \ldots)$ with respect to C′ is

$$xx_2-h(x+x_2)+r^2 = 0,$$

and these are identical if $\dfrac{x_1}{x_2-h} = \dfrac{r^2}{hx_2-r^2}$,

or $$hx_1x_2-r^2(x_1+x_2)+r^2h = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)},$$

which proves (1). At an intersection $hx-r^2 = 0$, and

$$(x-x_1)^2+y^2+z^2+\ldots+(x-x_2)^2+y^2+z^2+\ldots$$
$$\equiv 2(x^2+y^2+z^2+\ldots)-2x(x_1+x_2)+x_1^2+x_2^2$$
$$= 2r^2-\frac{2r^2}{h}(x_1+x_2)+x_1^2+x_2^2 = 2\frac{r^2}{h}(h-x_1-x_2)+x_1^2+x_2^2$$
$$= x_1^2+x_2^2-2x_1x_2 \text{ by (A)} \equiv (x_1-x_2)^2 \text{ which proves (2).}$$

[The above question has been generalised from Mr. McCay's, which gave the theorem for a *circle*, and of which a *geometrical* solution has been given on p. 41 of our Vol. xxxiv.]

3228 & 9098. (Professor Wolstenholme, M.A., Sc.D.)—(3228.) 1. In a tetrahedron ABCD, if AB be perpendicular to CD, and AC to BD, so also will AD be perpendicular to BC. Such a tetrahedron may be called rectangular, and we shall have

$$BC^2+AD^2 = CA^2+BD^2 = AB^2+CD^2.$$

2. In a rectangular tetrahedron, the perpendiculars from the angular points on the opposite faces meet in a point, the centre of perpendiculars. Such a point does not exist for any other tetrahedron.

3. The three straight lines, each intersecting at right angles a pair of opposite edges, meet in the centre of perpendiculars.

4. There exists a polar sphere to any rectangular tetrahedron, that is, each angular point is the pole of the opposite face with respect to a sphere; the centre of this sphere is the centre of perpendiculars. The sphere will not be possible unless all the plane angles, containing one of the solid angles, are obtuse.

5. One sphere can be drawn through the middle points of the six edges, and through the feet of the shortest distances between opposite edges, and the centre of this sphere is the centre of inertia of the tetrahedron, which bisects the distance between the centre of the circumscribed sphere and the centre of perpendiculars. (First twelve-points sphere.)

6. The sphere which passes through the centres of inertia of the faces will also pass through their centres of perpendiculars; and these points in any face will be the ends of a diameter of the circle in which the face cuts the sphere. This sphere also trisects the part of any perpendicular of the tetrahedron intercepted between the centre of perpendiculars and the corresponding angular point. (Second twelve-points sphere.)

7. These spheres and the circumscribed sphere have a common radical plane, and to this system of spheres belongs also the sphere of which the centre of inertia and the centre of perpendiculars are ends of a diameter.

8. If R, ρ be the radii of the circumscribed sphere and the polar sphere, and δ the distance between their centres, $\delta^2 = R^2 + 3\rho^2$; and the radius of the first twelve-points sphere is $\frac{1}{4}(R^2-\rho^2)^{\frac{1}{2}}$, that of the second is $\frac{1}{2}R$, and that of the sphere in (7) is $\frac{1}{4}(R^2+3\rho^2)^{\frac{1}{2}}$. The radius of the common section of all these spheres is $\frac{\rho}{\delta}(R^2-\rho^2)^{\frac{1}{2}}$.

9. If R, ρ' be the radii of the circumscribed and first twelve-points spheres, and δ' the distance between their centres, $\delta'^2 = R^2-3\rho'^2$.

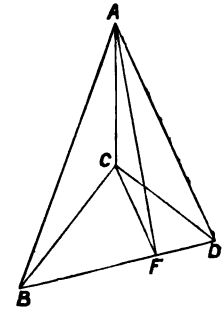
10. The centre of similarity of the circumscribed and second twelve-points sphere is the centre of perpendiculars of the tetrahedron; whence, if $ALA'a$ be drawn through the centres of perpendiculars to meet the opposite face in A′ and the circumscribed sphere in a, $La = 3LA'$.

9098. (W. J. C. Sharp, M.A.)—If a tetrahedron be such that the sums of the squares of its opposite edges are all equal, show that the nine-point circles of the triangles, which are its faces, are all sections of the same sphere. Also show, in space of n dimensions, that if (r, s) denote the line joining the r^{th} and s^{th} vertices of a simplicissimum, and if $A_1, A_2 \ldots A_{n+1}$ be $(n+1)$ magnitudes, such that $(r, s)^2 = A_r + A_s$ for all the values of r and s, the nine-point circles of all the triangles, which can be formed so as to have their vertices at three of those of the simplicissimum (see Quest. 8242), lie upon the same hyper-sphere.

Solution.

(3228). If AF be drawn perpendicular to BD, BD is perpendicular to AF and to the parallel to AC through F and so to the plane AFC,

and therefore CF is perpendicular to BD, and the perpendiculars upon the edges from the opposite angles of the faces which pass through them meet in the same point. Also



$$CD^2 - DF^2 = BC^2 - BF^2 = CF^2,$$

and $$AD^2 - DF^2 = AB^2 - BF^2 = AF^2,$$

$$\therefore \quad AB^2 + CD^2 = BC^2 + AD^2,$$

which proves (1); similarly, since AB is perpendicular to CD, $BC^2 + AD^2 = AC^2 + BD^2$,

And hence we may suppose

$$AB^2 = A_1 + A_2, \quad AC^2 = A_1 + A_3,$$
$$AD^2 = A_1 + A_4, \quad BC^2 = A_2 + A_3,$$
$$BD^2 = A_2 + A_4, \quad CD^2 = A_3 + A_4,$$

a form under which the tetrahedron is presented in a paper "On the Properties of Simplicissima, &c.," in the *Proc. of the Lond. Math. Soc.*, Vol. XVIII., pp. 357, 358, where the properties (2), (4), (5), and (7) are given as belonging to the corresponding class of simplicissima in space of n dimensions [those in which, the edge joining the r^{th} and s^{th} vertices being denoted by (r, s), $(r, s)^2 = A_r + A_s$, A_1, A_2, &c. being known magnitudes].

Taking the tetrahedron as tetrahedron of reference, the coordinates being tetrahedral, I have shown in the same paper that, if

$$S \equiv AB^2 . xy + AC^2 . xz + AD^2 . xw + BC^2 . yz + BD^2 . yw + CD^2 . zw,$$

$S = 0$ is the equation to the circumscribed sphere,

and that the perpendicular from A upon $x = 0$ is

$$(AB)^2 (x+y+z+w) - \frac{dS}{dy} = (AC)^2 (x+y+z+w) - \frac{dS}{dz}$$
$$= (AD)^2 (x+y+z+w) - \frac{dS}{dw},$$

that from B upon $y = 0$,

$$(AB)^2 (x+y+z+w) - \frac{dS}{dx} = (BC)^2 (x+y+z+w) - \frac{dS}{dz}$$
$$= (BD)^2 (x+y+z+w) - \frac{dS}{dw},$$

and so on, and if those equations be simultaneous

$$AC^2 - BC^2 = AD^2 - BD^2, \text{ &c., which proves (2).}$$

Also in the triangle AFC, the perpendiculars are those from A and C on the opposite faces, and the line which is perpendicular to BD and AC, and which therefore passes through the centre of perpendiculars of the tetrahedron, which coincides with that of ACF.

If $AB^2 = A_1 + A_2$, $AC^2 = A_1 + A_3$, $AD^2 = A_1 + A_4$, $BC^2 = A_2 + A_3$, $BD^2 = A_2 + A_4$, $CD^2 = A_3 + A_4$, the equations (given above) to the perpendicular from A upon $x = 0$ reduce to $A_2 y = A_3 z = A_4 w$, and so for the others, and at the centre of perpendiculars $A_1 x = A_2 y = A_3 z = A_4 w$. If there be a polar sphere, its equation

$$S - (x+y+z+w)(B_1 x + B_2 y + B_3 z + B_4 w) = 0$$

must reduce to $B_1x^2+B_2y^2+B_3z^2+B_4w^2=0$,

and therefore $B_1+B_2=A_1+A_2$, &c., &c.,

and $B_1=A_1$, $B_2=A_2$, &c.,

and the sphere is $S-(x+y+z+w)(A_1x+A_2y+A_3z+A_4w)=0$,

and at its centre (the pole of infinity $x+y+z+w=0$),

$$B_1x=B_2y=B_3z=B_4w, \quad \text{or} \quad A_1x=A_2y=A_3z=A_4w,$$

and the centre is the centre of perpendiculars. (This property, which is not noticed in the paper quoted, is true for all simplicissima of the kind treated of in note, p. 358.) Also, that the sphere may be possible, one at least of the quantities A_1, A_2, A_3, A_4 must be negative. Now

$$2A_1=AB^2+AC^2-BC^2=AC^2+AD^2-CD^2=AD^2+AB^2-BD^2,$$

and therefore, if A_1 be negative, all the angles at A are obtuse, and therefore (4). The quadric

$$A_1x^2+A_2y^2+A_3z^2+A_4w^2-(A_1+A_2)\,xy-(A_1+A_3)\,xz$$
$$-(A_1+A_4)\,xw-(A_2+A_3)\,yz-(A_2+A_4)\,yw-(A_3+A_4)\,zw=0,$$

which cuts the edges at the middle points and at the feet of the perpendiculars upon the edges from the opposite vertices, is a sphere, for the equation may be put into the form

$$S-\tfrac{1}{2}(x+y+z+w)(A_1x+A_2y+A_3z+A_4w)=0,$$

and it cuts the faces in their nine-point circles (9098), and similarly in higher space. At the centre

$$4A_1x-A_1(x+y+z+w)=4A_2y-A_2(x+y+z+w)$$
$$=4A_3z-A_3(x+y+z+w)=4A_4w-A_4(x+y+z+w),$$

and therefore $x=y=z=w$, or the centre is the centre of inertia.

Also the centre of perpendiculars, the centre of this sphere, and the centre of the circumscribed sphere are projected on any face into the centre of perpendiculars of the face, the centre of the nine-point circle, and the centre of the circumscribed circle, which lie in one line which is bisected by the centre of the nine-point circle, and therefore the centre of inertia is the middle point of the line joining the centre of the circumscribed sphere and the centre of perpendiculars (4).

If $S-(G_1x+G_2y+G_3z+G_4z)(x+y+z+w)=0$

be the sphere through the centroids of the faces $(0, \frac{1}{3}c, \frac{1}{3}c, \frac{1}{3}c)$, $(\frac{1}{3}c, 0, \frac{1}{3}c, \frac{1}{3}c)$, $(\frac{1}{3}c, \frac{1}{3}c, 0, \frac{1}{3}c)$, and $(\frac{1}{3}c, \frac{1}{3}c, \frac{1}{3}c, 0)$, [where $x+y+z+w\equiv c$],

$$2(A_2+A_3+A_4)=3(G_2+G_3+G_4), \text{ &c.,}$$

and $G_1=\frac{2}{3}A_1$, $G_2=\frac{2}{3}A_2$, $G_3=\frac{2}{3}A_3$, $G_4=\frac{2}{3}A_4$,

and the sphere is $S-\frac{2}{3}(A_1x+A_2y+A_3z+A_4w)(x+y+z+w)=0$;

where this meets the face $x=0$,

$$(A_2+A_3)yz+(A_3+A_4)zw+(A_4+A_2)wy-\tfrac{2}{3}(A_2y+A_3z+A_4w)(y+z+w)=0,$$

and this is satisfied if $A_2y=A_3z=A_4w$,

i.e., by the centre of perpendiculars of the face.

Also at the centre of the face

$$(A_2+A_3)\,z+(A_4+A_2)\,w-\tfrac{2}{3}A_2(y+z+w)$$
$$=(A_3+A_4)\,w+(A_2+A_3)\,y-\tfrac{2}{3}A_3(y+z+w)$$
$$=(A_4+A_2)\,y+(A_3+A_4)\,z-\tfrac{2}{3}A_4(y+z+w),$$

which reduce (since $y+z+w=c$) to

$$A_2(c-6y) = A_3(c-6z) = A_4(c-6w),$$

which are satisfied by the coordinates of the middle point of the line joining the centroid, and the centre of perpendiculars of the face

$$0,\quad \tfrac{1}{8}c\cdot\frac{\frac{4}{A_2}+\frac{1}{A_3}+\frac{1}{A_4}}{\frac{1}{A_2}+\frac{1}{A_3}+\frac{1}{A_4}},\quad \tfrac{1}{8}c\cdot\frac{\frac{1}{A_2}+\frac{4}{A_3}+\frac{1}{A_4}}{\frac{1}{A_2}+\frac{1}{A_3}+\frac{1}{A_4}},\quad \tfrac{1}{8}c\cdot\frac{\frac{1}{A_2}+\frac{1}{A_3}+\frac{4}{A_4}}{\frac{1}{A_2}+\frac{1}{A_3}+\frac{1}{A_4}},$$

and these are therefore the ends of a diameter of the circle in which the face cuts the sphere.

Again, where the perpendicular from A upon $x=0$,

$$A_2y = A_3z = A_4w \equiv \mu', \text{ suppose,}$$

cuts this sphere, $S-\frac{2}{3}(A_1x+A_2y+A_3z+A_4w)(x+y+z+w)=0$,

$$\tfrac{1}{3}(A_1x)\{y+z+w-2x\}+\mu'w=0,$$

or, omitting the value $x=0$, $A_1(c-3x)+3\mu'=0$;

and $3\mu'=2\mu$, therefore $x=\frac{1}{3}\left(\frac{2\mu}{A_1}+c\right)$,

or the line from the centre of perpendiculars to A is trisected by the sphere (since at the centre of perpendiculars $A_1z=A_2y=$ &c.), which proves (6).

From the equations to the three spheres already determined it is evident that they and the circumscribed sphere $S=0$ have a common radical plane $A_1x+A_2y+A_3z+A_4w=0$,

which is the plane in which the faces and edges of the tetrahedron whose vertices are at the feet of the perpendiculars meet the corresponding faces and edges of the original tetrahedron (p. 342).

If $S-\lambda(A_1x+A_2y+A_3z+A_4w)(x+y+z+w)=0$

be another sphere of the system which passes through the centre of inertia $x=y=z=w$,

$$3(A_1+A_2+A_3+A_4)-4\lambda(A_1+A_2+A_3+A_4)=0, \text{ and } \lambda=\tfrac{3}{4},$$

and its equation is

$$A_1x(y+z+w)+A_2y(z+w+x)+A_3z(w+x+y)+A_4w(x+y+z)$$
$$-\tfrac{3}{4}(A_1x+A_2y+A_3z+A_4w)(x+y+z+w)=0,$$

which is satisfied if $A_1x=A_2y=A_3z=A_4w$,

i.e., at the centre of perpendiculars.

At the centre of this sphere

$$A_1(y+z+w)+A_2y+A_3z+A_4w-\tfrac{3}{4}A_1(x+y+z+w)$$
$$=A_2(z+w+x)+A_1x+A_3z+A_4w-\tfrac{3}{4}A_2(x+y+z+w)=\text{&c.},$$

or $A_1(c-8x)=A_2(c-8y)=A_3(c-8z)=A_4(c-8w)=-4\mu$ suppose;

therefore $x=\frac{1}{2}\left(\frac{c}{4}+\frac{\mu}{A_1}\right)$, $y=\frac{1}{2}\left(\frac{c}{4}+\frac{\mu}{A_2}\right)$,

$z=\frac{1}{2}\left(\frac{c}{4}+\frac{\mu}{A_3}\right)$, $w=\frac{1}{2}\left(\frac{c}{4}+\frac{\mu}{A_4}\right)$,

and $$\frac{\mu}{A_1}+\frac{\mu}{A_2}+\frac{\mu}{A_3}+\frac{\mu}{A_4}=-c+2(x+y+z+w)=c;$$

therefore (x, y, z, w) is the middle point of the line joining the centre of perpendiculars and the centre of inertia (7).

In the same paper, p. 344, it appears that the equation to a sphere, radius ρ and centre (x', y', z', w'), is

$$S-(x+y+z+w)\frac{1}{c}\left\{x\frac{dS'}{dx'}+y\frac{dS'}{dy'}+z\frac{dS'}{dz'}+w\frac{dS'}{w'}\right.$$
$$\left.-\frac{c^2\rho^2+S'}{c}(x+y+z+w)\right\}=0,$$

and, if (x', y', z', w') be the point $\left(\frac{\mu}{A_1}, \frac{\mu}{A_2}, \frac{\mu}{A_3}, \frac{\mu}{A_4}\right)$ above

$$S'=3\mu c \quad \text{and} \quad \frac{dS'}{dx'}=A_1c+2\mu, \quad \frac{dS'}{dy'}=A_2c+2\mu, \text{ \&c.};$$

therefore $$\frac{1}{c}\left\{\frac{dS'}{dx'}-\frac{c^2\rho^2+S'}{c}\right\}$$
$$=\frac{1}{c}\left\{A_1c+2\mu-\frac{c^2\rho^2+3\mu c}{c}\right\}=\frac{1}{c}\left\{A_1-c\rho^2-\mu\right\},$$

which reduces to A_1 if $c\rho^2=-\mu$.

Now, by the same paper, $S'=c^2(R^2-\delta^2)$;

therefore $-3\rho^2=R^2-\delta^2$ or $\delta^2=R^2+3\rho^2$.

In the same way, if x'', y'', z'', ω'' be the point, $\frac{1}{4}c, \frac{1}{4}c, \frac{1}{4}c, \frac{1}{4}c$ the centre of the first twelve-point sphere, and ρ' its radius,

$$\frac{1}{c}\left\{\frac{dS''}{dx'}-\frac{c^2\rho'^2+S''}{c}\right\}=\tfrac{1}{2}A_1, \text{ \&c.}$$

Now $$S''=3(A_1+A_2+A_3+A_4)\tfrac{1}{16}c^2,$$

and $$\frac{dS''}{dx''}=(3A_1+A_2+A_3+A_4)\tfrac{1}{4}c;$$

therefore $$\rho'^2=\tfrac{1}{16}(A_1+A_2+A_3+A_4);$$

also $$S''=c^2(R^2-\delta'^2) \quad \text{or} \quad 3\rho'^2=R^2-\delta'^2 \ (9),$$

also, by (5), δ' is equal to the distance from (x', y', z', w'), the centre of perpendiculars, to (x'', y'', z'', w''), the centre of inertia, and therefore (*Proceedings of the London Mathematical Society*, Vol. XVIII., p. 344),

$$c^2\delta'^2=-S'-S''+x'\frac{dS''}{dx''}+y'\frac{dS''}{dy''}+z'\frac{dS''}{dz''}+w'\frac{dS''}{dw''}$$
$$=3c^2\rho^2-3c^2\rho'^2+\frac{\mu c}{4}\left\{\frac{1}{A_1}(3A_1+A_2+A_3+A_4)+\frac{1}{A}(A_1+3A_2+A_3+A_4)\right.$$
$$\left.+\frac{1}{A_3}(A_1+A_2+3A_3+A_4)+\frac{1}{A_4}(A_1+A_2+A_3+3A_4)\right\}$$
$$=3c^2(\rho^2-\rho'^2)+2\mu c+\frac{\mu c}{4}\left(\frac{1}{A_1}+\frac{1}{A_2}+\frac{1}{A_3}+\frac{1}{A_4}\right)(A_1+A_2+A_3+A_4)$$
$$=3c^2(\rho^2-\rho'^2)-2c^2\rho^2+4c^2\rho'^2=c^2(\rho^2+\rho'^2);$$

therefore $\delta'^2 = \rho^2 + \rho'^2 = R^2 - 3\rho'^2$; therefore $\rho'^2 = \frac{1}{4}(R^2 - \rho^2)$, (8).

Since $\frac{1}{2}\delta$ is the radius of the sphere in (7), its radius $= \frac{1}{4}(R^2 + 3\rho^2)^{\frac{1}{2}}$.

If a be the radius of the common section of the circum-sphere and the polar sphere, and b its distance from the centre of the latter,

$$a^2 + b^2 = \rho^2, \qquad a^2 + (\delta - b)^2 = R^2,$$

and therefore

$$a^2 = \frac{\rho^2}{\delta^2}(R^2 - \rho^2).$$

The tetrahedron, the vertices of which are at the points of trisection of the parts of the perpendiculars intercepted between the centre of perpendiculars and the angular points, is similar to the original tetrahedron, and all lines in it are $\frac{1}{3}$ of the corresponding lines in that, and hence the radius of its circumsphere (the second twelve-point sphere) is $\frac{1}{3}R$. And the centre of perpendiculars is one centre of similarity, so that, as the second twelve-point sphere passes through A′ (the centre of perpendiculars of the face opposite to A), $La = 3LA'$, (10).

I had unfortunately not seen this interesting Question when I prepared the paper so often cited; in fact, it is only now (January, 1888) that it has been communicated to me by the kindness of Mr. Miller, the Editor. If I had seen the Question, I should have added generalisations of several parts of it to the note mentioned (p. 358). I think it best now to give these in the form of a question, which will hereafter appear in the *Educational Times*.

(9098). If a simplicissimum in space of n dimensions be such that $A_1, A_2 \ldots A_{n+1}$ being fixed magnitudes, and (r, s) denoting the edge joining the r^{th} and s^{th} vertices, $(r, s)^2 = A_r + A_s$ for all values of r and s (different); not only are the nine-point circles of the triangles, whose vertices coincide with any three of the vertices of the simplicissimum, sections of the same spheric (hyper-sphere) (A), but also the perpendiculars from the vertices on the opposite faces meet in a point, which is the centre of the self-conjugate spheric (B). Another spheric (C) passes through the centroids of the faces and their centres of perpendiculars. All these spherics have a common radical linear surface with the circumspheric, which is the locus upon which the linear loci through the vertices of the original simplicissimum meet those through the corresponding vertices of the simplicissimum whose vertices are the feet of the perpendiculars upon the opposite faces. And another spheric (D) through the centre of perpendiculars and the centroid has the same radical locus with the circumspheric. The spherics (A) and (C) become identical in space of two dimensions. Also the spheric (C) cuts off $1/n$ of the portions of the perpendiculars intercepted between the centre of perpendiculars and the faces. If L be the centre of perpendiculars, AA′ the perpendicular from A upon the opposite face, and a the point where ALA′a meets the circumspheric, $La = n\text{LA}'$.

[These Solutions show that the idea of the simplicissimum not only gives a clearer idea of higher space, but also assists in the conception of the relations of space of two and three dimensions, and it is likely to lead to some interesting results in ordinary geometry.]

2081. (The late Professor Rutherford, LL.D.)—A triangle A′B′C′ has its angles respectively complementary to the half angles of the tri-

angle ABC, and its side B'C' equal to BC; prove that

$$\Delta A'B'C' : \Delta ABC = \sin \tfrac{1}{2}A : 2 \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C.$$

Solution.

Let A' be the centre of the escribed circle opposite to the triangle, then A'BC is the triangle A'B'C'; and if p_1 be the perpendicular from A on BC, and S the area of ABC,

$$\begin{aligned}\Delta A'BC : \Delta ABC &= r_1 : p_1 = \frac{S}{S-a} : \frac{2S}{a} = a : b+c-a \\ &= \sin A : \sin B + \sin C - \sin A \\ &= 2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A : 4 \cos \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C \\ &= \sin \tfrac{1}{2}A : 2 \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C.\end{aligned}$$

9120. (Professor KALIPADA BASU, M.A.)—If p and q are integers, show that

$$1 - \frac{q}{p+1} + \frac{q(q-1)}{(p+1)(p+2)} - \frac{q(q-1)(q-2)}{(p+1)(p+2)(p+3)} + \ldots + (-1)^q \frac{q(q-1)(q-2)\ldots 2.1}{(p+1)(p+2)\ldots(p+q)} = \frac{p}{p+q}.$$

Solution.

The formula holds for all values of p if $q = 1$. Assume it to hold for $q-1$ and $p+1$, so that

$$1 - \frac{q-1}{p+2} + \frac{(q-1)(q-2)}{(p+2)(p+3)} + \ldots + (-1)^{q-1} \frac{(q-1)(q-2)\ldots 1}{(p+2)(p+3)\ldots(p+q)} = \frac{p+1}{p+q},$$

$$\therefore \quad 1 - \frac{q}{p+1} + \frac{q(q-1)}{(p+1)(p+2)} - \ldots + (-1)^q \frac{q(q-1)(q-2)\ldots 1}{(p+1)(p+2)\ldots(p+q)}$$

$$\equiv 1 - \frac{q}{p+1}\left\{1 - \frac{q-1}{p+2} + \frac{(q-1)(q-2)}{(p+2)(p+3)} - \ldots + (-1)^{q-1} \frac{(q-1)(q-2)\ldots 1}{(p+2)(p+3)\ldots(p+q)}\right\}$$

$$= 1 - \frac{q}{p+1} \cdot \frac{p+1}{p+q} = \frac{p}{p+q},$$

and therefore, if the formula holds for $q-1$ and $p+1$, it holds for q and p; but it holds for 1 and any value of p, and therefore for 2 and any value of p, and so on, for all integral values of q.

6903. (W. J. C. SHARP, M.A.)—Find the law of density, varying as the depth, that the centre of pressure of a semicircle, just immersed

with its diameter horizontal, may divide the vertical radius as $m : n$.

Solution.

If p be the pressure at any point of the fluid, ρ the density, and x the depth below the surface, $\rho = \frac{dp}{dx}$; and if $\bar{x}$ be the depth of the centre of pressure of the plane surface bounded by any curve, the axes being rectangular, and that of x vertical,

$$\bar{x} \int y\, p\, dx = \int xy\, p\, dx,$$

the integrals being taken between the proper limits; in this case

$$\bar{x} \int_0^a y\, p\, dx = \int_0^a xy\, p\, dx,$$

where a is the radius of the semicircle.

Let $yp \equiv \phi''(x)$; then, since $\bar{x} = \frac{m}{m+n} a$,

$$\frac{m}{m+n} a \int_0^a \phi''(x)\, dx = \int_0^a x\phi''(x)\, dx,$$

$$\frac{ma}{m+n} \{\phi'(a) - \phi'(0)\} = a\phi'(a) - \phi(a) + \phi(0)$$
$$- (m/n + 1)\, \phi(a) + a\phi'(a) + m/n\, l + (m/n + 1)\, l' = 0,$$

therefore $$a^{-(m/n+1)}\, \phi(a) - la^{-m/n} - l' a^{-(m/n+1)} = l',$$

and $$\phi(x) = lx + l' + l'' x^{m/n+1},$$

therefore $$yp = \phi''(x) = kx^{m/n-1},$$

and $$p = \frac{kx^{(m-n)/n}}{(a^2 - x^2)^{\frac{1}{2}}} \quad \text{or} \quad k \frac{x^{(m-n)/n}}{(2ax - x^2)^{\frac{1}{2}}},$$

according as the semicircle is placed with the horizontal diameter above or below, and therefore

$$\rho = k \frac{d}{dx} \left\{ \frac{x^{(m-n)/n}}{(a^2 - x^2)^{\frac{1}{2}}} \right\} \quad \text{or} \quad k \frac{d}{dx} \left\{ \frac{x^{(m-n)/n}}{(2ax - x^2)^{\frac{1}{2}}} \right\}.$$

[Of this Question three unsatisfactory solutions have been already published in our Volumes. The solution given on p. 94 of Vol. XLIV. solves the related problem obtained by writing, in the Question, centre of gravity for centre of pressure. The Question is an example of what may be called inverse problems of centre of gravity, &c., an interesting class which may be found useful in practical investigation, and to which belongs Question 9434.]

6392. (E. B. ELLIOTT, M.A.)—A_1, A_2, A_3, A_4 are the vertices of a tetrahedron, P is the point whose volume coordinates with regard to it (*i.e.*, the ratios $PA_2A_3A_4 : A_1A_2A_3A_4$, &c.) are x_1, x_2, x_3, x_4, and O is any other point. Prove that, a_{12} being the length of the edge A_1A_2,

$$OP^2 = x_1 OA_1^2 + x_2 OA_2^2 + x_3 OA_3^2 + x_4 OA_4^2 - \Sigma (a_{12}^2 x_1 x_2).$$

Solution.

This question is a case of the more general one, if $A_1, A_2, \ldots A_{n+1}$ be the vertices of a simplicissimum in space of n dimensions, P a point whose content coordinates are x', y', z' ..., and O any other point,

$$OP^2 = \frac{1}{V}\{x' . OA_1^2 + y' . OA_2^2 + \ldots\} - \frac{1}{V^2}\Sigma a_{12}^2 x'y',$$

or

$$x' . OA_1^2 + y' . OA_2^2 + \ldots - \Sigma a_{12}^2 x'y',$$

according as x', y' ... stand for the simplicissima $PA_2A_3 \ldots A_{n+1}$, &c., or for their ratios to the simplicissimum of reference.

If the coordinates have the first signification, I have shown (*Proc. Lond. Math. Soc.*, Vol. XVIII., pp. 344—46) that

$$S \equiv \Sigma a_{12}^2 x'y' = 0$$

is the equation to the circumscribed spheric, and that, if d be the distance between two points $(x, y, \ldots)$ and $(x_1, y_1, \ldots)$,

$$V^2 d^2 = -S - S_1 + \left(x_1 \frac{dS}{dx} + y_1 \frac{dS}{dy} + \&c.\right).$$

Therefore, if O be $(x, y, \ldots)$, and A_1 $(V, 0, 0\ldots)$, since $S = 0$ at this point,

$$V^2 . OA_1^2 = -S + V\frac{dS}{dx},$$

similarly,

$$V^2 . OA_2^2 = -S + V\frac{dS}{dy}, \quad \&c. \quad \&c.$$

If then P be the point $(x', y' \ldots)$,

$$V^2\{x' . OA_1^2 + y' . OA_2^2 + \ldots\} = -S(x' + y' + \ldots) + V\left(x'\frac{dS}{dx} + y'\frac{dS}{dy} + \&c.\right)$$

$$= -S . V + V\left(x'\frac{dS}{dx} + y'\frac{dS}{dy} + \&c.\right) \text{ (for } x' + y' + \ldots = V),$$

but

$$V^2 . OP^2 = -S - S' + \left(x'\frac{dS}{dx} + y'\frac{dS}{dy} + \&c.\right),$$

therefore

$$V^3 . OP^2 = V^2\{x' . OA_1^2 + y' . OA_2^2 + \ldots\} - VS',$$

or

$$OP^2 = \frac{1}{V}\{x' . OA_1^2 + y' . OA_2^2 + \ldots\} - \frac{1}{V^2}S',$$

which gives the other form at once if x', y', &c. be written for $\frac{x'}{V}$, $\frac{y'}{V}$, &c.

2109. (Prof. WOLSTENHOLME, M.A., Sc.D.)—If A_1, A_2, A_3, A_4, A_5 be five fixed points in space of three dimensions, P any other point, V_1, V_2, ... the volumes of the tetrahedra $A_2A_3A_4A_5$, &c., taken of such signs that

$$V_1 + V_2 + V_3 + V_4 + V_5 = 0,$$

prove that (1) $V_1 . PA_1^2 + V_2 . PA_2^2 + V_3 . PA_3^2 + V_4 . PA_4^2 + V_5 . PA_5^2$ will be independent of the position of P; and (2) if C_1, C_2, C_3, C_4, C_5 be the centres of the spheres circumscribing the tetrahedra $A_2A_3A_4A_5$, &c., r_1, r_2, r_3, r_4, r_5 the radii of these spheres,

$$\frac{1}{C_1A_1^2 - r_1^2} + \frac{1}{C_2A_2^2 - r_2^2} + \frac{1}{C_3A_3^2 - r_3^2} + \frac{1}{C_4A_4^2 - r_4^2} + \frac{1}{C_5A_5^2 - r_5^2} = 0,$$

$$V_1(C_1A_1^2 - r_1^2) = V_2(C_2A_2^2 - r_2^2) = \ldots = V_1 . PA_1^2 + \ldots .$$

[The corresponding result for two dimensions, at least the equation $\frac{1}{C_1A_1^2 - r_1^2} + \ldots = 0$, was set in the Mathematical Tripos Examination for 1886; the proof is precisely the same for both, and is a rather nice example of the use of determinants.]

Solution.

I have shown (*Proceedings of the London Mathematical Society*, Vol. XVIII., pp. 337, 338) that, if (x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2), (x_3, y_3, z_3, w_3), (x_4, y_4, z_4, w_4) be the tetrahedral coordinates of the vertices of a tetrahedron, V the volume of the tetrahedron of reference (therefore $x + y + z + w = V$), the volume

$$= \frac{1}{V^3} \begin{vmatrix} x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \\ z_1, & z_2, & z_3, & z_4 \\ w_1, & w_2, & w_3, & w_4 \end{vmatrix}$$

If, then, $A_1A_2A_3A_4$ be taken as tetrahedron of reference, and A_5 be (x_1, y_1, z_1, w_1), by the question,

$$0 = \frac{1}{V^3} \begin{vmatrix} 1, & 1, & 1, & 1, & 1 \\ V, & 0, & 0, & 0, & x_1 \\ 0, & V, & 0, & 0, & y_1 \\ 0, & 0, & V, & 0, & z_1 \\ 0, & 0, & 0, & V, & w_1 \end{vmatrix} \quad \ldots\ldots\ldots\

therefore $V^2\{V_1 . PA_1^2 + V_2 . PA_2^2 + V_3 . PA_3^2 + V_4 . PA_4^2 + V_5 . PA_5^2\}$

$$= \frac{1}{V^3}\begin{vmatrix} V^2 . PA_1^2, & V^2 . PA_2^2, & V^2 . PA_3^2, & V^2 . PA_4^2, & V^2 . PA_5^2 \\ V, & 0, & 0, & 0, & x_1 \\ 0, & V, & 0, & 0, & y_1 \\ 0, & 0, & V, & 0, & z_1 \\ 0, & 0, & 0, & V, & w_1 \end{vmatrix}$$

$$= \frac{1}{V^3}\begin{vmatrix} V\frac{dS}{dx}, & V\frac{dS}{dy}, & V\frac{dS}{dz}, & V\frac{dS}{dw}, & -S_1 + x_1\frac{dS}{dx} + y_1\frac{dS}{dy} + z_1\frac{dS}{dz} + w_1\frac{dS}{dw} \\ V, & 0, & 0, & 0, & x_1 \\ 0, & V, & 0, & 0, & y_1 \\ 0, & 0, & V, & 0, & z_1 \\ 0, & 0, & 0, & V, & w_1 \end{vmatrix}$$

by (I.)

$$= \frac{1}{V^3}\begin{vmatrix} 0, & 0, & 0, & 0, & -S_1 \\ V, & 0, & 0, & 0, & x_1 \\ 0, & V, & 0, & 0, & y_1 \\ 0, & 0, & V, & 0, & z_1 \\ 0, & 0, & 0, & V, & w_1 \end{vmatrix}$$ which is independent of the position of P

$\equiv K$, suppose.

If P coincide with C_1, $PA_2 = PA_3 = PA_4 = PA_5 = r_1$,

and $$V_1 . C_1A_1^2 + V_2 . r_1^2 + V_3 . r_1^2 + V_4 . r_1^2 + V_5 . r_1^2 = \frac{K}{V^2},$$

and $$V_1 . r_1^2 + V_2 . r_1^2 + V_3 . r_1^2 + V_4 . r_1^2 + V_5 . r_1^2 = 0,$$

therefore $$V_1(C_1A_1^2 - r_1^2) = \frac{K}{V^2} = V_2(C_2A_2^2 - r_2^2) = \&c.$$

$$= V_1 . PA_1^2 + V_2 . PA_2^2 + V_3 . PA_3^2 + V_4 . PA_4 + V_5 . PA_5^2,$$

and $$0 = V_1 + V_2 + V_3 + V_4 + V_5$$

$$= \frac{K}{V^2}\left\{\frac{1}{C_1A^2 - r_1^2} + \frac{1}{C_2A_2^2 - r_2^2} + \frac{1}{C_3A_3^2 - r_3^2} + \frac{1}{C_4A_4^2 - r_4^2} + \frac{1}{C_5A_5^2 - r_5^2}\right\},$$

and therefore, &c.

This proof would apply equally if $V_1, V_2, \ldots V_{n+2}$ were the contents of the simplicissima whose vertices are situated at all $(n+2)$-points $A_1, A_2, \ldots A_{n+2}$, except A_1, A_2, &c. respectively, and hence for the four triangles as stated in the note.

9093 & 9170. (Samuel Roberts, F.R.S.)—(9093.) Four spheres are bisected by the faces of a tetrahedron, and each of them passes through the three vertices on the corresponding face. Show that the sphere

cutting these four spheres orthogonally has for its centre the isogonal conjugate of the centre of the circumscribing sphere (*i.e.*, if α, β, γ, δ are the quadruplanar coordinates of the centre of the circumscribing sphere, $1/\alpha$, $1/\beta$, $1/\gamma$, $1/\delta$ are those of that of the orthogonal sphere), and that the sphere through the centres of the four spheres first mentioned has its centre at the middle point between those of the orthogonal and circumscribing spheres, and passes through the common section of those spheres. Compare the result with an analogous property of the nine-point circle in plane space.

(9170). Show that, (1) if the sides of a triangle (produced if necessary) are cut by a circle, and on the linear intercepts as diameters three circles are drawn, the radical centre of these is the "isogonal conjugate," relative to the triangle, of the centre of the first-named circle; (2) if the planes of the faces of a tetrahedron are cut by a sphere, and on the circular intercepts as diametral sections four spheres are constructed, the radical centre of these spheres is the "isogonal conjugate" relative to the tetrahedron of the centre of the first-named sphere.

["Isogonal conjugates" are points whose trilinear or quadruplanar coordinates are proportional to (α, β, γ), $(1/\alpha, 1/\beta, 1/\gamma)$, or $(\alpha, \beta, \gamma, \delta)$, $(1/\alpha, 1/\beta, 1/\gamma, 1/\delta)$. These theorems may be obtained geometrically.]

Solution.

The very interesting properties of triangles and tetrahedra enunciated in these Questions are true for all simplicissima, and the following solution proceeds as if the word simplicissimum, which includes triangles, tetrahedra, &c., were substituted for tetrahedron in the questions.

In a paper read before the London Mathematical Society ("On Simplicissima, &c.," *Proceedings*, Vol. XVIII., pp. 325—59), I have shown that in space of n-dimensions, if the coordinates λ, μ, ν ... of any point P be contents of the simplicissima which have a common vertex at P and stand upon the faces of the simplicissimum of reference—the content of which is V, and those of its faces V_1, V_2 ...—the equation to the spheric (hypersphere) whose centre is $(\lambda', \mu' \ldots)$ and radius ρ is

$$S - \frac{\lambda+\mu+\ldots}{V}\left(\lambda\frac{dS'}{d\lambda'}+\mu\frac{dS'}{d\mu'}+\ldots\right)+\frac{V^2\rho^2+S'}{V^2}(\lambda+\mu+\ldots)^2=0,$$

where $$S \equiv \lambda\mu\,(1\,.\,2)^2+\mu\nu\,(2\,.\,3)^2+\nu\lambda\,(1\,.\,3)^2+\ldots,$$

[(1 . 2) denoting the edge joining the vertices opposite to $\lambda=0$ and $\mu=0$]; that $S=0$ is the equation to the circumspheric; that, if R be its radius and $(\lambda', \mu' \ldots)$ its centre,

$$\frac{dS'}{d\lambda'}=\frac{dS'}{d\mu'}=\ldots=2VR^2 \quad \text{and} \quad S'=V^2R^2;$$

and that, if d denote the distance between any two points $(\lambda', \mu' \ldots)$, $(\lambda'', \mu'' \ldots)$,

$$V^2d^2=-S'-S''+\left(\lambda'\frac{dS''}{d\lambda''}+\mu'\frac{dS''}{d\mu'}\ldots\right).$$

Now, if $$\Sigma \equiv S-\frac{\lambda+\mu+\ldots}{V}\left(\lambda\frac{dS'}{d\lambda'}+\mu\frac{dS'}{d\mu'}\ldots\right)+\frac{V^2\rho^2+S'}{V^2}(\lambda+\mu\ldots)^2=0$$

be the equation to a spheric, centre $(\lambda', \mu' \ldots)$ radius ρ,

$$\Sigma-k_1\lambda\,(\lambda+\mu\ldots)=0, \quad \Sigma-k_2\mu\,(\lambda+\mu\ldots)=0, \quad \&c.$$

will be the equations to spherics cutting $\lambda=0$, $\mu=0$, &c., respectively, in sections identical with those made by $\Sigma=0$; and the radical centre of these spherics is $k_1\lambda = k_2\mu = \dots$;

and if R_1 be the radius and $(\lambda_1, \mu_1 \dots)$ the centre of $\Sigma - k_1\lambda(\lambda+\mu\dots)=0$,

i.e., $$S-\frac{\lambda+\mu\dots}{V}\left\{\lambda\left(\frac{dS'}{d\lambda'}+k_1V\right)+\mu\frac{dS'}{d\mu'}+\nu\frac{dS'}{d\nu'}\right\}+\frac{V^2\rho^2+S'}{V^2}(\lambda+\mu\dots)^2=0,$$

this must be identical with

$$S-\frac{\lambda+\mu\dots}{V}\left\{\lambda\frac{dS_1}{d\lambda_1}+\mu\frac{dS_1}{d\mu_1}+\dots\right\}+\frac{V^2R_1^2+S_1}{V^2}(\lambda+\mu\dots)^2=0,$$

and therefore

$$\left.\begin{aligned}
\frac{dS'}{d\lambda'}+k_1V-\frac{V^2\rho^2+S'}{V}&=\frac{dS_1}{d\lambda_1}-\frac{V\,R_1^2+S_1}{V}\\
\frac{dS'}{d\mu'}-\frac{V^2\rho^2+S'}{V}&=\frac{dS_1}{d\mu_1}-\frac{V^2R_1^2+S_1}{V}\\
\frac{dS'}{d\nu'}-\frac{V^2\rho^2+S'}{V}&=\frac{dS_1}{d\nu_1}-\frac{V^2R_1^2+S_1}{V}\\
\text{&c.}\qquad&\qquad\text{&c.}
\end{aligned}\right\}\quad\dots\dots\dots\dots\dots\dots\text{(A)},$$

Multiplying these equations by λ', μ' ... respectively, and adding,

$$2S'+k_1V\lambda'-V^2\rho^2-S'=\left(\lambda'\frac{dS_1}{d\lambda_1}+\mu'\frac{dS_1}{d\mu_1}+\dots\right)-V^2R_1^2-S_1,$$

therefore $$k_1V\lambda' = -S'-S_1+\left(\lambda'\frac{dS_1}{d\lambda_1}+\mu'\frac{dS_1}{d\mu_1}+\dots\right)+V^2(\rho^2-R_1^2)$$
$$= V^2d_1^2+V^2(\rho^2-R_1^2),$$

if d_1 is distance between $(\lambda', \mu' \dots)$ and $(\lambda_1, \mu_1 \dots)$; but, if $\lambda_1=0$ (as in the question),

$$d_1^2=\rho^2-R_1^2=\left(\frac{n\lambda'}{V_1}\right)^2,$$

therefore $k_1V\lambda' = 2V^2d_1^2$, and $k_1 = 2V\frac{n^2\lambda'^2}{V_1^2}\cdot\frac{1}{\lambda'} = 2n^2\frac{V}{V_1^2}\lambda'$,

and at the radical centre of the spherics $(\lambda'', \mu'' \dots)$,

$$k_1\lambda''=k_2\mu''=\dots,\quad\text{therefore}\quad\frac{\lambda'\lambda''}{V_1^2}=\frac{\mu'\mu''}{V_2^2}=\text{&c.},$$

or

$$\frac{\lambda''}{V_1}:\frac{\mu''}{V_2}:\dots=\frac{V_1}{\lambda'}:\frac{V_2}{\mu'}:\dots;$$

and the points are isogonal conjugates; for, if $(\alpha', \beta' \dots)$, $(\alpha'', \beta'' \dots)$ be the perpendiculars from them on the faces of the simplicissimum of reference,

$$\alpha'=\frac{n\lambda'}{V_1},\quad\beta'=\frac{n\mu'}{V_2},\quad\text{&c.},$$

and therefore

$$\alpha'':\beta'':\dots=\frac{1}{\alpha'}:\frac{1}{\beta'}:\dots,$$

The first part of 9093 follows at once (for the radical centre whose spherics is the centre of the orthogonal spheric) if $\Sigma \equiv S$.

Let $(\lambda''', \mu''', \ldots)$ be the centre, and ρ the radius of the spheric through the centres of the sections by the faces. Its equation is

$$S - \frac{\lambda + \mu + \ldots}{V}\left(\lambda''' \frac{dS}{d\lambda} + \mu''' \frac{dS}{d\mu} + \ldots\right) + \frac{V^2\rho^2 + S'''}{V^2}(\lambda + \mu \ldots)^2 = 0,$$

and this passes through the points $(0, \mu_1, \nu_1 \ldots)$, $(\lambda_2, 0, \nu_2 \ldots)$, &c., which satisfy the equations (A), when R is written for ρ. From these, since

$$\frac{dS'}{d\lambda'} = \frac{dS'}{d\mu'} \ldots = 2VR^2,$$

$$\frac{dS_1}{d\lambda_1} - k_1 V = \frac{dS_1}{d\mu_2} = \frac{dS_1}{d\nu_2} = \ldots = \frac{dS'}{d\lambda'} - VR^2 - VR^2 + VR_1^2 + \frac{S_1}{V}$$

$$= VR_1^2 + \frac{S_1}{V} = 2VR_1^2,$$

since $2S_1 = \left(\lambda_1 \frac{dS_1}{d\lambda_1} + \mu_1 \frac{dS_1}{d\mu_1} + \ldots\right) = (\mu_1 + \nu_1 + \ldots) \times \left(VR_1 + \frac{S_1}{V}\right)$

$$= V R_1^2 + S_1;$$

but, since $(0, \mu_1 \ldots)$ lies on the spheric above,

$$S_1 - \left(\lambda''' \frac{dS_1}{d\lambda_1} + \mu''' \frac{dS_1}{d\mu_1} + \ldots\right) + V^2\rho^2 + S''' = 0,$$

or $$V^2R_1^2 - 2V^2R_1^2 - k_1 V\lambda''' + V^2\rho^2 + S''' = 0,$$

therefore $$k_1 V\lambda''' = S''' - S_1 + V^2\rho^2.$$

Similarly, $$k_2 V\mu''' = S''' - S_2 + V^2\rho^2, \text{ &c.},$$

but $$k_1 V\lambda' = 2V^2 d_1^2 = 2V^2(R^2 - R_1^2) = 2S' - 2S_1,$$

therefore $k_2 V\mu' = = 2S' - 2S_2,$

&c. &c. &c.;

therefore $k_1 V(2\lambda''' - \lambda') = 2S''' - 2S' + 2V^2\rho^2 = k_2 V(2\mu''' - \mu') = \text{&c.}$

From these equations, $V\left\{2(\lambda''' + \mu''' + \ldots) - (\lambda' + \mu' + \ldots)\right\}$

$$\equiv V^2 = 2(S''' - S' + V^2\rho^2)\left(\frac{1}{k_1} + \frac{1}{k_2} + \ldots\right);$$

but $$k_1\lambda'' = k_2\mu'' = \ldots = q \text{ suppose},$$

therefore $$V(= \lambda'' + \mu'' + \ldots) = q\left(\frac{1}{k_1} + \frac{1}{k_2} + \ldots\right),$$

therefore $$Vq = 2(S''' - S' + V^2\rho^2),$$

and $$2\lambda''' - \lambda' = \lambda'', \quad 2\mu''' - \mu' = \mu'', \text{ &c.},$$

and $(\lambda''', \mu''' \ldots)$ is the mid-point of the line $(\lambda', \mu' \ldots)$, $(\lambda'', \mu'' \ldots)$. In space of two dimensions the spheric (circle) whose centre is $(\lambda''', \mu''' \ldots$ is the nine-point circle, since it is the circle through the mid-points of the sides. (This spheric also passes through the projections of the conjugate point, *i.e.*, in space of two dimensions, the feet of the perpendiculars.)

If r be the radius of the orthogonal sphere, its equation is

$$\sigma \equiv S - \frac{\lambda+\mu+\dots}{V}\left(\lambda''\frac{dS}{d\lambda}+\mu''\frac{dS}{d\mu}+\dots\right)+\frac{V^2r^2+S''}{V^2}(\lambda+\mu+\dots)^2 =$$

also $$S \equiv S - \frac{\lambda+\mu+\dots}{V}\left(\lambda'\frac{dS}{d\lambda}+\mu'\frac{dS}{d\mu}+\dots\right)+\frac{V^2R^2+S'}{V^2}(\lambda+\mu+\dots)^2 = 0,$$

therefore $$S+\sigma \equiv 2S - \frac{\lambda+\mu+\dots}{V}\left((\lambda'+\lambda'')\frac{dS}{d\lambda}+(\mu'+\mu'')\frac{dS}{d\mu}+\dots\right)$$

$$+\frac{V^2(r^2+R^2)+S'+S''}{V^2}(\lambda+\mu+\dots)^2$$

$$\equiv 2\left\{S - \frac{\lambda+\mu+\dots}{V}\left(\lambda'''\frac{dS}{d\lambda}+\mu'''\frac{dS}{d\mu}+\dots\right)+\frac{V^2\rho^2+S'''}{V^2}(\lambda+\mu+\dots)^2\right\}$$

$$+V^2(r^2+R^2)+S'+S''-2V^2\rho^2-2S'''$$

$$= 2\left\{S - \frac{\lambda+\mu+\dots}{V}\left(\lambda'''\frac{dS}{d\lambda}+\mu'''\frac{dS}{d\mu}+\dots\right)+\frac{V^2\rho^2+S'''}{V^2}(\lambda+\mu+\dots)^2\right\},$$

for $$V^2(r^2+R^2)+S'+S''-2V^2\rho^2-2S'''$$

$$= V^2\left(r^2+R_1^2+\frac{n^2\lambda'^2}{V_1^2}\right)-(2S'''-2S'+2V^2\rho^2)+S''-S'$$

$$= -S''-S_1+\lambda''\frac{dS_1}{d\lambda_1}+\mu''\frac{dS_1}{d\mu_1}+\dots+V^2\frac{n^2\lambda'^2}{V_1^2}-k_1V\lambda''+S''-S'$$

$$= -V^2R_1^2+2V^2R_1^2+V^2\frac{n^2\lambda'^2}{V_1^2}-V^2R^2 = V^2\left(\frac{n^2\lambda'^2}{V_1^2}+R_1^2-R^2\right) = 0,$$

and therefore the three spherics have a common section.

APPENDIX IV.

NOTE ON PERFECT NUMBERS,*

By R. W. D. Christie.

[*These numbers have engaged the attention of mathematicians from very early times, but there are still several problems connected with them requiring a solution before the subject can be said to be fully elucidated. Under date Dec. 20th,* 1638, Descartes *wrote as follows:—"Et je ne sais pourquoi vous jugez qu'on ne saurait parvenir par ce moyen à l'invention d'un vrai nombre parfait; que si vous en avez une demonstration, j'avoue qu'elle est au delà de ma portée et que je l'estime extrêmement; car, pour moi, je juge qu'on peut trouver des nombres impairs véritablement parfait."*

A perfect number is one which is equal to the sum of all the numbers which divide it without remainder.†

Let $y^n x$ be a perfect number. Then we have

$$y^n x = 1 + y + y^2 \ldots y^n + x(1 + y + y^2 \ldots y^{n-1}),$$

provided x be a prime number.

Therefore
$$y^n x = \frac{y^{n+1}-1}{y-1} + \frac{y^n-1}{y-1}x$$
$$= \frac{y^{n+1}+y^n x - x - 1}{y-1};$$

therefore
$$x = \frac{y^{n+1}-1}{y^{n+1}-2y^n+1}.$$

If we make the denominator of this fraction equal to unity, we shall ensure x being an integer; therefore assume $y^{n+1} = 2y^n$, then $y = 2$, and $y^n x = 2^n(2^{n+1}-1) = P$, a perfect number, provided $2^{n+1}-1 = x$ be a prime, *i.e.*, provided $n+1$ be a prime.‡

From this value of P we can make several deductions.

I. *Every perfect number equals a sequence from unity, but from no other number.*

For every sequence from unity is of form $\frac{m \,.\, (m+1)}{2}$.

Therefore, by choosing a suitable value for m, we can make this $= P$.

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* My apology for venturing on the subject is the paucity of information respecting these numbers in the ordinary text-books on Algebra.

† Various definitions have been given, but the above does not exclude unity.

‡ *Any* prime will not do, and it is an interesting problem what prime p will make 2^p-1 a prime.

Let $$m = 2^{n+1} - 1,$$

then $$\frac{m+1}{2} = 2^n,$$

therefore $$P = \frac{m \cdot m+1}{2} = 2^n (2^{n+1} - 1) = \text{a sequence from unity.}$$

If P be equal to any other sequence, we shall have

$$\mathrm{P} = a + a + 1 + a + a + \ldots + a + (n-1)$$

$$= m\left\{a + \frac{m-1}{2}\right\} = 2m_1\left\{1 - a + \frac{2m_1 - 1}{2}\right\},$$

where m and m_1 are *odd factors* of P and also the number of terms of the sequence; a being the first term of the sequence.

Now, as perfect numbers consist of the product of an even number (viz. 2^n) and a prime $(2^{n+1}-1)$, there are only two possible *odd factors*, viz., unity and the prime. Consequently, there can only be two possible sequences formed from a perfect number, viz., the number itself, and a sequence from unity, obtained by giving m the two values of unity and the prime.

Thus a is either P or unity.

If $P = 1$, the sequence is either 1 or 0 + 1.

II. *Every perfect number* (with one exception) *is the product of a square and a prime.*

Since every prime is odd, except the number 2, therefore n is always even; therefore $P = 2^n (2^{n+1} - 1)$ = a square × a prime.

The single exception to this (caused by the exceptional prime 2) is $P = 6$.

In the case of unity we have $1 \times 1 = P$.

III. Let $N = a^p b^q c^r \ldots$, where $a, b, c \ldots$ are successive primes.

Then the *number* of factors is

$$n = (p+1)(q+1)(r+1) \ldots;$$

and the *sum* of the factors is

$$s = \frac{a^{p+1} - 1}{a-1} \cdot \frac{b^{q+1} - 1}{b-1} \cdot \frac{c^{r+1} - 1}{c-1} \cdots$$

Now in the case of perfect numbers these equations become

$$P = 2^n \cdot b^1,$$

therefore $$n = 2(n+1) = \text{number of factors,}$$

and $$s = (2^{n+1} - 1)(b+1).$$

These equations include the number itself.

If we separate the odd and the even factors, we readily obtain the following equations:—

(1) Number of *odd* factors = 2
(2) Number of *even* factors = $2n$
(3) Total number of factors = $2(n+1)$

} inclusive of P,

and

(4) Number of *odd* factors $= 2$
(5) Number of *even* factors $= 2n-1$ } exclusive of P,
(6) Total number of factors $= 2n+1$

where $n+1$ in all cases is a prime.

Remembering that, when $P = y^n x = 1$, n is zero, and that 1 is a square number with an occult factor as usual, all the above equations are satisfied in the case of unity.

Similarly for the *sums* of the factors we get

(1) Sum of *odd* factors $= 2^{n+1}$
(2) Sum of *even* factors $= 2(2^n-1)(2^{n+1})$ } inclusive of P,
(3) Total number of factors $= (2^{n+1}-1)(2^{n+1})$

and

(4) Sum of *odd* factors $= 2^{n+1}$
(5) Sum of *even* factors $= 2^n(2^{n+1}-3)$ } exclusive of P.
(6) Total number of factors $= 2^n(2^{n+1}-1) = P$

We thus see that, if $P = y^n x$, we have

x = sum of factors of y^n,
y^n = half sum of factors of x,
or = half sum of odd factors of P.

IV. *Perfect numbers* (with same exception) *are of form* $9N+1$.

For $2^n/9$ always has remainders 1, 4, 7, recurring for any integral value of n.

And, if $2^n/9$ leaves remainders 1, 4, 7,

then $2^{n+1}-1/9$ will leave remainders 1, 7, 4 ;

therefore $P = 2^n(2^{n+1}-1) + 9$ will always have unity as a remainder

Thus P, with one exception, is of form $9N+1$.

V. *Perfect numbers end in the digits* 6 *or* 8.

For 2^n, since n is even with one exception, always ends in the digits 4 or 6.

And $2^{n+1}-1$, a prime, always ends in 1 or 7.

Also, when 2^n ends in 4, then $(2^{n+1}-1)$ ends in 7.

And, when 2^n ends in 6, then $(2^{n+1}-1)$ ends in 1.

Therefore $P = 2^n(2^{n+1}-1)$ always ends in 6 or 8.

Thus perfect numbers are even, except in the doubtful case of unity.

VI. *The product of the factors of a perfect number is*

$$\{2^n(2^{n+1}-1)\}^n = P^n, \text{ where } n+1 \text{ is a prime.}$$

The factors of (2^n) are $1, 2^1, 2^2, 2^3 \ldots\ldots 2^n$,

the factors of $(2^{n+1}-1)$ are 1 and $2^{n+1}-1$;

therefore the factors of $P = 1, 2^1, 2^2 \ldots 2^n$ and $(2^{n+1}-1)(1, 2^1, 2^2 \ldots 2^{n-1})$.

Thus the product of $P = 2^n(2^{n+1}-1).2^n(2^{n+1}-1).2^n(2^{n+1}-1)\ldots$ to n terms

$$= \{2^n(2^{n+1}-1)\}^n.$$

And as n is even in all cases, except one, the product must be a square.

VII. *The number of positive integers less than a perfect number, and prime to it, is* $2^n(2^n-1)$.

Let $N = a^p \cdot b^q \cdot c^r \ldots\ldots$, where a, b, c are successive primes.

Then $$\phi(N) = N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots\ldots$$

This equation in the case of a perfect number becomes

$$\phi(P) = P(1-\tfrac{1}{2})\left(1-\frac{1}{2^{n+1}-1}\right) = 2^n(2^n-1);$$

thus $\phi(P)/P = (2^n-1)/(2^{n+1}-1)$, which is $< \frac{1}{2}$.

VIII. *The sum of all the integers which are less than P, and prime to it, is*

$$= \frac{P}{2}\,\phi(P) = (2^n-1)(2^{n+1}-1)(2^{2n-1}).$$

(Unity is considered prime to all numbers.)

IX. If d_1, d_2, d_3 ... denote the divisors of any number N,

$$N = \phi(d_1) + \phi(d_2) + \phi(d_3)\ldots\ldots$$

with perfect numbers, this becomes

$$P = \phi(d_1) + \phi(d_2) + \phi(d_3) + \phi(d_4) + \phi(d_5) + \&c. \ldots\ldots\ldots\ldots (1).$$

Also, from the definition, $P = d_1 + d_2 + d_3 + d_4 + \&c.$ ..... (2).

These two equations are equal to the following

$$P = 1+1+2^1+2^2\ldots+2^{n-1}+(1+1+2^1+2^2+\ldots+2^{n-1})(2^{n+1}-2)$$
$$= 2^n+(2^n)(2^{n+1}-2) = 2^n(2^{n+1}-1) \ldots\ldots\ldots\ldots (1).$$

And $$= 1+2^1+2^2+2^3\ldots 2^n+(1+2^1+2^2+\ldots 2^{n-1})(2^{n+1}-1)$$
$$= 2^{n+1}-1+(2^n-1)(2^{n+1}-1) = 2^n(2^{n+1}-1) \ldots\ldots\ldots\ldots (2).$$

Example: $496 = 1+1+2+4+8+30+30+60+120+240$ ..... (1).

$$= 1+2+4+8+16+31+62+124+248\ldots\ldots\ldots\ldots (2).$$

X. *List of Perfect Numbers.*

Value of n.	*Numbers.*	*Factors.*	*Remainder after casting out Nines.*
1*	6	3 × 2†	6
		2 × 3‡	
2	28	4 × 7	1
4	496	= 16 × 31	1
6	8128	= 64 × 127	1
12	33550336	= 4096 × 8191	1
16	8589869056	= 65536 × 131071	1
18	137438691328	= 262144 × 524287	1
30	2305843008139952128	= 1073741824 × 2147483647	1
	&c.	&c.	

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* p_e^n equals a prime only with these even indices with one exception, viz., $p = 2$, $n = 1$.

† From $y^n x$ when $y = 3$ and $x = 2$ in the original equation.

‡ From $y^n x$ when $y = 2$ and $x = 3$ in the original equation.

Many interesting problems may be formed from the above data; *e.g.*,—

(1) If p = any perfect number, and π = any prime number (except 5), show that $p^4 \sim \pi^4$ is a multiple of 5.

(2) Find a number which is = sum of its factors.
= product of its factors.
= its totient.
= product of the totients of its components.

(3) Is unity a true perfect number? and are there other perfects?

(4) Prove that, if $P = a^p b^q$ be a binary perfect number odd or even, then $a_s^p b_s^q / a^p b^q$ minus unity = $(a_s^{p-1} b^q + b_s^{q-1} a^p + a_s^{p-1} b_s^{q-1}) / a^p b^q$ = unity.

Secondary Perfects.—In the strict sense of the word there is no perfect number. If we make $y = 3$ in the original equation, we can obtain integral values for x, but only when n = zero and unity.

The equation then is $x = (3^{n+1} - 1)/(3^n + 1)$,

$y = 4$ gives us $x = (4^n - 1)/(2^{2n-1} + 1)$,

and so on for higher values of y, but the equations lead to no useful result.

If P_2 = a three-fold perfect number = $y^n xz$, suppose, we get

$$xz/(x + z + 1) = (y^{n+1} - 1)/\{(y-2)(y^n) + 1\};$$

and if $y = 2$ we get $xz/x + z + 1 = 2^{n+1} - 1$ from which to get integral values of x and z.

If $P_2 = y^n xz$ = a perfect whose factors = $3P$, we get, if $y = 2$,

$$xz/x + z + 1 = (2^{n+1} - 1)/(2^n + 1),$$

which is possible in integers.

Thus, let $x = 3$, $z = 5$, $n = 3$; then

$$P_2 = y^n xz = 2^3 . 3 . 5 = 120.$$

Other examples of this secondary class of perfects are:—

$$P_2 = 2^5 . 3 . 7 = 672, \qquad P_2 = 2^9 . 3 . 11 . 31 = 523776,$$

$$P_2 = 2^{13} . 3 . 11 . 43 . 127, \quad P_2 = 2^8 . 5 . 7 . 19 . 37 . 73,$$

and consequently, $P_4 = 2^8 . 3 . 5 . 7 . 19 . 37 . 73.$

Other perfects equal to four times their factors are

$$P_4 = 2^8 . 3^3 . 5 . 7, \quad P_4 = 2^7 . 3^6 . 5 . 17 . 23 . 137 . 547 . 1093.$$

&c. &c.

It will be noticed that the values of n in 2^n in these secondary perfects are just those that are not available in P_2 or the ordinary perfects.

Some call *Secondary Perfects* those which are equal to the *product* of their aliquot parts, as *e.g.*, $27 = 1 \times 3 \times 9$; $14 = 1 \times 2 \times 7$. These two classes are comprised in the formulæ (*a*) $P = p^n$, where p is any prime and $n = 3$; (*b*) $P = a^p b^q$, where a, b are primes, and p, q = unity.

Odd Perfects.—With regard to *odd* perfects the question is, Can we solve the equation

$$2 = (a_s^p b_s^q c_s^r \ldots)/(a^p b^q c^r \ldots),$$

where $a, b, c \ldots$ are odd primes, and a_s^p means the sum of the factors of a^p?

(1) It is clear that only one of the components $a_\alpha^p b_\beta^q \ldots$ can be even, we should not be able to cancel out. Let us suppose a_α^p even, and to contain no other power of 2 except $2'$.

A careful study of the factors of p_α^n, where p is any prime, shows us that q, r, s ... must be even, and both a and p of the form $4n+1$. The indices however cannot all be *four*, for p_α^4 is prime to primes not ending in unity (except to 5 when p is of form $5x+1$), The following example illustrates this :—

Let $N = a^p b^q c^r \ldots = 7^1 . 3^4 . 11^2 . 19^1$, suppose;

then $2 = 8/7 \times 121/81 \times 133/121 \times 20/19 \ldots$

$= 160/81$, a result slightly too small.

Let us add another component to N, 71^1 say;

then $2 = 160/81 \times 72/71 = 1280/639$, a result slightly too large;

and so we might go on *ad inf.* adding component after component, but we should not be able to cancel out the *even* numerator out of the *odd* denominator. It should be carefully remembered that the numbers in the denominator are all odd primes or powers of primes.

(2) It is easy to see that the smallest prime admissible for a is 5; but I think it can be shown that this number is inadmissible in any odd perfect, and, if this be so, we are practically restricted in attempting to solve the above equation to the use of numbers in the numerator ending in the digits 1, 3, and 7, and in the denominator to square numbers ending in unity and 9 or else unity alone, of course excepting one component which provides the desired result, viz., 2. For, as we shall see, p_α^n ends in 5 only when n ends in 4 and p in 1.

E.g., let $N = 5' . b^q . c^r \ldots\ldots$,

then $2 = 6/5 \times b_\beta^q / b^q \times c_\alpha^r / c^r \ldots$,

or $1 = 3/5 \times b_\beta^q / b^q \times c_\alpha^r / c^r \ldots$.

To solve the equation we must cancel out the 5. Therefore b must be some prime ending in unity raised to the $2(5m-3)^{\text{th}}$ power. Let us take the most favourable case, and assume $a^p = 5^1$ instead of 5^9 (5^5 being inadmissible) or any higher index (for both must be of form $4n+1$); for, if we are unable to solve the equation with the former, *à fortiori* with the latter. For before we could cancel even 5^2 we should be obliged to use 11^{24} (or 31^{24} as the case may be), and there would still remain 5^7 to cancel out of some other prime ending in unity, say 41 raised to the 64th power at least, and then our troubles of cancellation with 11^{24} and 41^{64} would begin.

Our equation, then, now stands

$$1 = \frac{3}{5} \times \frac{16105}{11^4} \times \frac{c_\alpha^r}{c^r} \ldots,$$

$$= 3 \times \frac{3221}{11^4} \times \frac{c_\alpha^r}{c^r} \ldots.$$

The number 3 must now be got rid of. This makes $c^r = 3^2$ (at least),

therefore, $$1 = \frac{3}{1} \times \frac{3221}{11^4} \times \frac{13}{3^2} \ldots$$

$$= \frac{3221}{11^4} \times \frac{13}{3} \ldots.$$

In attempting to cancel out the primes remaining in the numerator (not to mention those in the denominator) we shall find we have gone far beyond our tether, and the equation is impossible, the right-hand side of the equation having become greater than unity. It must not be forgotten that any component c_σ^r/σ^r is a fraction > than unity, decreasing in value as c increases, but increasing as the index increases, so that there is a limit in either direction. Had we used 5^9, we should still have had to cancel the 3 in addition to other difficulties.

The next higher prime admissible is 13, and by a similar exhaustive process (did space permit) we could show that this prime compels the use of the component $\frac{57}{7^2}$ $\left(\text{or } \frac{2801}{7^4}, \text{ which is worse}\right)$, which again compels the use of $\frac{381}{19^2}$ (or something much worse), which again compels the use of a component or components so large as to make the equation impossible. Similarly for any higher prime.

As aids to a solution of this *palmary* theorem* (as Professor SYLVESTER terms it), perhaps the following results may be of use:—

Let $p_\sigma^n = (p^{n+1}-1)/(p-1)$ = sum of factors of p^n when p is any prime.

(A.) *On the digits of* p_σ^n.

(1) p_σ^n ends in $(n+1)$ when p ends in unity, therefore

(a) p_σ^n ends in 5 or 9 only when n ends in 4 or 8 and p ends in 1.

(b) p_σ^{20} can end in unity only.

(2) p_σ^{4n} ends in unity and $4n+1$ only, *i.e.*,—

(a) In unity when p ends in 3, 7, or 9,

(b) In $4n+1$ when p ends in unity.

(c) p_σ^4 is only divisible by primes in unity (except by 5 when p ends in unity).

(3) p_σ^{4n-2} ends in 1, 3, or 7:

(a) In 3 when p ends in 3.

(b) In 7 when p ends in 7.

(c) In 1 when p ends in 9.

(B.) *On the factors of* p_σ^n.

a_σ^n is a factor of p_σ^n when p is of form $a_\sigma^n x + a^n$.

Ex. gr.: 7 is a factor of 23_σ^2, since 23 is of form $7x+2$.

31 is a factor of 97_σ^4, since 97 is of form $31x+2^n$.

&c. &c.

(C.) *On the equation* $p_\sigma^n \mathrm{M} = p_\sigma^{m(n+1)-1}$, where m and n are any integers, and $m > 1$.

Ex. gr.: (a) Let $p = 2$, then $2_\sigma^n M = 2_\sigma^{m(n+1)-1}$.

In the case of perfect numbers M and m are both here = unity.

(b) Let $p = 5$, $n = 9$, $m = 2$. Then 5_σ^9 is a factor of 5_σ^{19}, &c. &c.

(c) This equation will assist in determining what values of p are to be rejected where $2^p - 1$ = a prime.

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* "There seems," says Prof. SYLVESTER, "every reason to believe that Euclid's Perfect Numbers are the only Perfect Numbers which exist." (See *Nature, Dec. 15th*, 1887.)

(D.) *On the equation* $p_\sigma^{m^{n-1}} = \pi M$, *where* π *is the highest prime in* $p^m - 1/p - 1$ *and* p *is any prime.*

Let $m = 2$, and $p = 5$.

Then $5_\sigma^{2^{n-1}} = 6M$. This proves that, if 5 be a factor of any odd perfect, then (1) 3 must be a factor also.

Ex.: Let $n = 1$, then $5^1/5 = 6/5$,

$n = 3$, then $5_\sigma^5/5^5 = 3906/5^5 = 2 \cdot 3^2 \cdot 7 \cdot 31/5^5$,

(therefore this component is impossible, for 3, 5 and 7 cannot coexist)

$n = 5$, then $5_\sigma^9/5^9 = 2441406/5^9$, &c. &c.

(2) Some prime ending in unity necessarily accompanies 5.

(3) Since 5 and 3 necessarily go together, then 7 is necessarily discarded, for the three cannot coexist, since $\frac{6}{5} \times \frac{13}{3^2} \times \frac{57}{7^2}$ is > 2.

(4) Since $p_\sigma^{2^{n-1}} = (p+1)M$, therefore 3 appears as a factor wherever p is of form $3x+2$, and 7 appears when p is of form $7y+6$; therefore, if any prime is simultaneously of form $3x+2$ and $7y+6$, it cannot co-exist with 5; *e.g.*, 41 or any prime of form $42n-1$, such as 83, 167, 251, &c. Or, again.

(5) If p is of form $3x+1$, and simultaneously of form $7x+2^n$ $(n<3)$, then such elements as $37_\sigma^{3^{n-1}}/37^{3^{n-1}}$, 79, 163, &c., cannot co-exist with 5 for a similar reason.

(6) p_σ^2 is not a prime if p is of any of the forms $3x+1$, $7y+2^n$, $13z+3^n$, where $n < 3$, &c. &c.

(7) Similarly, if 13 is a factor of an odd perfect, then 7 must necessarily be, &c. &c.

(E.) (*a*) If p be anyprime of form $9m+1$, then p_σ^n is of form $n+1$ (mod. 9).

Ex. gr., 19_σ^2 is of form 3 (mod. 9); therefore no prime.

(*b*) If p be of form $9m+8$,

then p_σ^n when n is odd $= 0$ (mod. 9), therefore no prime;

then p_σ^n when n is even $= 1$ (mod. 9).

(*c*) Any other prime p of the $6M-1$ (or $2 \cdot 5 \cdot 8$ class),

then p_σ^n is of form 1, 4, or 7 (mod. 9).

(*d*) The fraction p_σ^n/p^n becomes when n is infinite $p/p-1$, and when n is unity $p+1/p$. Thus the difference between the greatest and least values of p_σ^n/p^n can never exceed $1/p\,(p-1)$.

(*e*) Similarly the difference between the greatest and least values of p^n/p_σ^n can never exceed $1/(p)(p+1)$.

(*f*) $p_\sigma^n = (p+1)^n - p\,(\times)$.

(*g*) p_σ^n is always prime to p^n.

p_σ^n is odd if n is even, and *vice versâ* when p is a prime > 2.

(*h*) 3_σ^n is always of form 4 (mod. 9).

p_σ^{2n} always ends in unity.

p^n can never be a perfect number since p_σ^{n-1}/p^n is always < 1.

When we consider how often p_σ^n is not a prime, and consequently splits up into two or more primes, compelling the use of fresh components

consisting of numbers in the numerator which, if primes themselves, are so large as to make the equation impossible, or if not primes split up into primes which in their turn compel the use of others *ad inf.* before they can be cancelled, the finding of an odd perfect appears a problem very improbable, if not impossible.

(F.) *On* 2^p-1, *when* p *is a prime.*

We have $2^p-1 = (1+1)^p-1 = px+1$.

If $x = pnr+n+r$. We get $2^p-1 = (pn+1)(pr+1)$.

Let $2^p-2 = M$.

Then $n = (M-r)(pr+1)$ to find even integral values of n and r, where M and p are given.

Ex. : $2^{11}-1 = (11n+1)(11r+1)$.

Therefore $n = (186-r)/(11r+1)$.

If $r = 2$, then $n = 8$.

Thus $2^{11}-1 = 23 \times 89$.

Similarly
$$2^{23}-1 = 47 \times 178481,$$
$$2^{29}-1 = 233 \times 2304167,$$
$$2^{37}-1 = 223 \times 616318177,$$
$$2^{43}-1 = 431 \times 20408568497, \text{ \&c. \&c.}$$

Thus since none of these values of p make 2^p-1 a prime, they are not factors of perfect numbers.

The equation $p_v^n = p_v^{n-m} + p^{n-m+1} \cdot p_v^{m-1} = p_v^{m-1} + p^m \cdot p_v^{n+m}$.

will assist in separating 2^p-1 into factors.

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(1) If $2^p-1 = P$ where p and P are primes, then P is of form $6N+1$.

(2) If p is of form $6N+1$, then P is of form $9N+1$.
If p is of form $6N-1$, then P is of form $9N+4$.

(3) 2^n-1 is always of form $4N+3$ for any integral value of n.

(4) 2^p-1 is of form $1+6p\,(M)$, where p is a prime.

APPENDIX V.

SOLUTIONS OF SOME OLD QUESTIONS,

By Robert W. D. Christie.

2814. (The late Matthew Collins, B.A.)—Can the common difference of three rational square integers in A. P. be ever equal to 17?

Solution.

Let $(a)^2$, $(a+d)^2$, $(a+2d)^2$ be the three squares in A. P.

Then $(a+d)^2-a^2 = (a+2d)^2-(a+d)^2$, or $d^2 = 3d^2$.

Thus there cannot be three rational square integers in A. P.

2912. (The Editor.)—Find three rational numbers such that, if their squares be diminished by 2, 3, 5 respectively, the three remainders shall be rational squares.

Solution.

Let x, y, z be the numbers. Then $x^2-2=(x-n)^2$ say. Or $x=(n^2+2)/2n$. Here n may be any integer, and, as the difference between two integral squares can never be 2, x is necessarily a fraction.

Similarly $y = (n^2+3)/2n$, $z = (n^2+5)/2n$.

Example: Let n = unity in each case, then the squares are $(\frac{3}{2})^2$, 2^2, 3^2 respectively.

The equation $X = (n^2+m)/2n$ is the general solution where $m = 2, 3, 5$ or any integer.

3836. (R. Tucker, M.A.)—The digits of a number of two figures are multiplied together, and if the resulting number consists of two figures these are also multiplied together, and the process continued until there is a single digit only; find the respective chances of this ultimate digit being (1) 5, or (2) 8, or (3) a square number; also (4) find the same of a number of three digits.

Solution.

There are 90 numbers of three digits.

Of these three, viz., 15, 35, and 75, with three more formed by reversal of digits, can produce 5; therefore chance for 5 is $\frac{1}{3}$.

We can produce 8 from the numbers 18, 24, and 36, and those formed from reversal of digits; together with others formed from the *factors* of these six, we get 22 in all. Thus the chance for 8 is $\frac{11}{45}$.

Similarly for 4 and 9 the chance is $\frac{2}{15}$.

The results for the three digits are almost similar.

It is curious to note that the chance for unity is the least of all, being $\frac{1}{90}$ and $\frac{1}{900}$, and for *zero* the greatest.

The chances for all the digits beginning with the least are in this order —1, 7, 3, 9, 5, 4, 2, 6, 8, 0.

7990. (Professor HANUMANTA RAU, M.A.)—Find the relation between the positive integers X, n, Y, such that the number $X^2 + nY^2$ may be split up into factors of the form $x^2 + ny^2$.

Solution.

Let $X^2 + nY^2 = (x^2 + ny^2)(r^2 + ns^2)$.

And assume $X^2 = x^2r^2 + n^2s^2y^2 = (mxr - nsy)^2$ say .....(1).

Then $nY^2 = n(y^2r^2 + s^2x^2) = n(pyr - sx)^2$ say .....(2),

therefore $2m/(m^2-1) = xr/nsy$ .....(1),

and $2p/(p^2-1) = yr/sx$ .....(2).

Assume $xr = 2m$ and $yr = 2p$.

Then $nsy = m^2 - 1$ and $sx = p^2 - 1$.

Therefore $X^2 + nY^2 = (m^2+1)^2 + n(p^2+1)^2 = (x^2 + ny^2)(r^2 + ns^2)$.

And $n = (2m)(m^2-1) / (2p)(p^2-1) = (m-1)(m)(m+1) / (p-1)(p)(p+1)$,

where m and p may be any integers whatever, provided n be any integer; to get which is easy, since it is equal to the quotient of two sequences.

Ex. gr.—Let $p = 2$, $m = 3$, $n = 4$ Then

$$X^2 + nY^2 = 10^2 + 4(5^2) = (3^2 + 4 \cdot 2^2)(2^2 + 4 \cdot 1^2).$$

8201. (R. TUCKER, M.A.)—Six numbers of three figures are formed by different arrangements of three unequal digits; find the digits when the numbers divided by 7 have only three different remainders.

Solution.

Let the digits be a, b, and c. Then the six numbers will be

$N_1 = 100a + 10b + c$; $N_2 = 100a + b + 10c$; $N_3 = 100b + 10a + c$;

$N_4 = b + 10a + 100c$; $N_5 = 100b + 10c + a$; $N_6 = 10b + 100c + a$.

We have then

$N_1 - N_2 = 9\,(b \sim c)$; $N_3 - N_4 = 99\,(b \sim c)$; $N_5 - N_6 = 90\,(b \sim c)$.

Let $(b \sim c) = 7$, as, *e.g.*, 8 and 1, or 9 and 2; then a may be any different number. There will then be three remainders only to modulus 7. *Ex. gr.*, take 129. The six numbers formed from these three digits will give remainders 2, 3, or 4, to modulus 7 (and for a similar reason the same remainder 3 to the three moduli 7, 8, or 9). The principle extends to larger numbers so long as the last three digits only vary.

8294. (D. Edwardes.)—Prove that the equation

$$(h^2 - ab)(ax^2 + 2hxy + by^2 + 2gx + 2fy) - af^2 - bg^2 + 2fgh = 0$$

represents a pair of straight lines, and show how they are connected with the conic given by the general equation.

Solution.

We have $$ax^2 + 2hxy + by^2 + 2gx + 2fy + \frac{c_1}{h_2 - ab} = 0,$$

where $$c_1 = (h^2 - ab)\,c = -af^2 - bg^2 + 2fgh;$$

therefore the discriminant $\Delta = (ab - h^2)\,c + 2fgh - af^2 - bg^2 = 0$

is the condition that the conic may be represented by two straight lines.

8297. (Piers C. Ward, B.A.)—Prove that, if

$$y^2 + yz + z^2 = a, \quad z^2 + zx + x^2 = b, \quad x^2 + xy + y^2 = c,$$

then
$$(a-b)(4b - 3x^2)^{\frac{1}{2}} \pm (a-c)(4c - 3x^2)^{\frac{1}{2}} = 3(b-c)x,$$
$$(b-c)(4c - 3y^2)^{\frac{1}{2}} \pm (b-a)(4a - 3y^2)^{\frac{1}{2}} = 3(c-a)y,$$
$$(c-a)(4a - 3z^2)^{\frac{1}{2}} \pm (c-b)(4b - 3z^2)^{\frac{1}{2}} = 3(a-b)z.$$

Solution.

We have $(2x + y)^2 = 4x^2 + 4xy + 4y^2 - 3y^2 = 4c - 3y^2$;

and $(2x + z)^2 = 4x^2 + 4xz + 4z^2 - 3z^2 = 4b - 3z^2$.

Similarly $2y + x = \pm(4c - 3x^2)^{\frac{1}{2}}$, and $2y + z = \pm(4a - 3z^2)^{\frac{1}{2}}$.

Also $2z + y = \pm(4a - 3y^2)^{\frac{1}{2}}$, and $2z + x = \pm(4b - 3x^2)^{\frac{1}{2}}$.

The desired result is now easily established.

8333. (Professor B. HANUMANTA RAU, B.A.)—Prove that the equations

$$x^5 + 19x - 40 = 0 \quad \text{and} \quad 7x^4 - 12x^3 + 4bx^2 + 12x + 7 = 0$$

have two common roots.

Solution.

Any biquadratic $x^4 + px^3 + qx^2 + rx + s = 0$ may be solved as a quadratic if $r^2 - p^2 s = 0$. Whence the given equation

$$= \left\{x^2 - \tfrac{4}{7}x - 1\right\}^2 = 0.$$

This gives us value of $b = -\frac{31}{14}$. Hence, by common algebraical work, we can reduce the two given equations to the ordinary cubic

$$y^3 + \tfrac{1560315}{819841}\, y = \tfrac{2767329290}{574805361},$$

from which y may be obtained at leisure.

Or we may take the G. C. M. of the two equations and equate it to zero, the roots of this equation will then coincide with the required roots which are common to the two equations.

8370. (Professor MUKHOPÂDHYÂY, M.A., F.R.S.E.)—A piece of cardboard is 8 inches by 12 inches. Find how many gun-wads $\frac{3}{4}$ inch in diameter can be punched out of it.

Solution.

The maximum number of gun-wads which can be punched

$$= (8 \times 12)/(\tfrac{3}{4} \times \tfrac{3}{4}) = 170\tfrac{2}{3}.$$

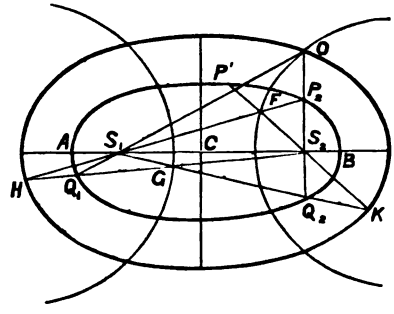
If we so plan the cardboard into sections, that the length and breadth of each shall be divisible by three, there will then be no loss of cardboard,

	in.	in.	sq. in.	
therefore	6	× 12	= 72 =	128 gun-wads,
	6	× 3	= 18 =	32 gun-wads,
	6	× 1	= 6 =	10 gun-wads.
			Total	170 gun-wads.

8413. (F. MORLEY, B.A.)—If OPSQ, $OP_1S_1Q_1$ are focal chords of an ellipse, show that the four points in which SP_1, SQ_1 meet S_1P, S_1Q, lie on the confocals through O, two on each.

Solution.

Taking C for origin, and $CS_1 = CS_2 = c = (a^2 - b^2)^{\frac{1}{2}}$,



equation of given ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ..... (1),

$x'y'$ = point O, $\left.\begin{aligned} \frac{y}{x+c} &= \frac{y'}{x'+c} \\ \frac{y}{x-c} &= \frac{y}{x'-c} \end{aligned}\right\}$ O $\left\{\begin{aligned} &\text{.....} \quad (2), \\ &\text{.....} \quad (3), \end{aligned}\right.$

x_1y_1 = point P_1, $\left.\begin{aligned} \frac{y}{x+c} &= \frac{y_2}{x_2+c} \end{aligned}\right\}$ F $\left\{\begin{aligned} &\text{.....} \quad (4), \\ &\text{.....} \quad (5), \end{aligned}\right.$

x_2y_2 = point P_2, $\left.\begin{aligned} \frac{y}{x+c} &= \frac{y_1}{x_1-c} \end{aligned}\right\}$

$\xi_1\eta_1$ = point Q_1, $\left.\begin{aligned} \frac{y}{x+c} &= \frac{\eta_2}{\xi_2+c} \end{aligned}\right\}$ G $\left\{\begin{aligned} &\text{.....} \quad (6), \\ &\text{.....} \quad (7). \end{aligned}\right.$

$\xi_2\eta_2$ = point Q_2, $\left.\begin{aligned} \frac{y}{x-c} &= \frac{\eta_1}{\xi-c} \end{aligned}\right\}$

Solve the pairs of equations indicated, to get the coordinates of F, G, H, K (8), and then obtain x_1, y_1, ξ_1, η_1, by solving (1) and (2), and x_2, y_2, ξ_2, η_2 from (1) and (3), and afterwards substitute the results in (8). The equations of the confocals through O are equated with the two values of λ given by

$$x^2/a^2 + \lambda + y^2/b^2 + \lambda = 1, \qquad x'^2/a^2 + \lambda + y'^2/b^2 + \lambda = 1,$$

The values in (8) must then satisfy these equations. The four points are F, G, H, K, *i.e.*, F, G on the hyperbola, and H, K on the ellipse, the two confocals through O at right angles to each other at O.

8457. (W. J. C. Sharp, M.A.)—If E be the spherical excess of a triangle whose sides are a, b, c; show (1) that

$$\sin^2 \tfrac{1}{4}E = \frac{\sin \frac{1}{2}s \sin \frac{1}{2}(s-a) \sin \frac{1}{2}(s-b) \sin \frac{1}{2}(s-c)}{\cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c},$$

where $2s = a+b+c$; and hence (2) deduce the ordinary formula for the area of a plane triangle.

Solution.

The first part of this Question is completely solved in TODHUNTER'S *Spherical Trigonometry*, 3rd edit., p. 72; and Art. 132 of the same book will solve the second part of the Question.

8459. (PIERS C. WARD, M.A.) — Prove, without expansion, the equality of the determinants

$$\begin{vmatrix} a+b+c & 3a & 3a \\ 3b & a+b+c & 3c \\ 3b & 3b & a+b+c \end{vmatrix} = \begin{vmatrix} a+b+c & 3c & 3c \\ 3b & a+b+c & 3c \\ 3b & 3b & a+b+c \end{vmatrix}.$$

Solution.

If the constituent $3c$ in the third column and second row of the first determinant were altered to $3a$, the determinants would then be equal, but they are not so as given.

8472. (Professor SIRCOM, M.A.)—Show that the value of x from the equation $10^x = x+1$ is $x = \cdot 13712916$.

Solution.

The given equation is incorrect.

For $10^x = 10^{\cdot 13712916}$ is *not* $= 1+x = 1\cdot 13712916$, but $= 1\cdot 3712916$.

8666. (R. TUCKER, M.A.)—Solve the equation

$$b^2x^3-(a^2-b^2)\,x^2+(a^2-c^2)\,x-c^2 = 0,$$

where a, b, c are sides of a triangle.

Solution.

This is a general cubic equation of the form x^3-px^2+qx-r, with the restriction that $p+1 = q+r$, and any two of the roots > the third. The solution in terms of p, q, and r leads to very complicated results, but by eliminating the second term, we get

$$y^3-y\left\{\tfrac{1}{3}(p^2-3q)\right\}-\tfrac{1}{27}(2p^3-9pq+27r) = 0 \quad \ldots\ldots\ldots\ldots\ldots(1),$$

where $p = (a^2-b^2)/b^2$, $q = (a^2-c^2)/b^2$, $r = c^2/b^2$,

also $$p+1 = q+r = a^2/b^2.$$

Equating (1) with $\cos^3 a - 3\cos a/4 - \cos 3a/4 = 0$,

we get for the three roots of the equation

$$2(\tfrac{1}{3}t)^{\frac{1}{2}}\cos a, \quad \text{and} \quad 2(\tfrac{1}{3}t)^{\frac{1}{2}}\cos(2\pi \pm a),$$

where $$t = (p^2-3q)/3, \quad \text{and} \quad \cos 3a = \tfrac{1}{2}s(3/t)^{\frac{3}{2}}.$$

The equation $s = 2t^{\frac{3}{2}}$ is the condition for equal roots, where

$$s = (2p^3 - 9pq + 27r)/27.$$

The three roots in Arithmetical Progression are

$$(a^2-b^2)/3b^2, \quad (a^2-b^2)/3b^2 \pm \left\{(a^4 - 5a^2b^2 + b^4 + 3b^2c^2)/3b^4\right\}^{\frac{1}{2}}.$$

If the roots are in Harmonical Progression, the equation

$$z = (a^2-c^2)^2 \pm \left\{(a^2-c^2)^4 - 27b^2c^4(a^2-c^2)\right\}^{\frac{1}{2}}/9b^2c^2$$

will give the greatest and least of them.

If $rp^3 = q^3$, *i.e.*, if $(a^2-c^2)/(a^2-b^2) = \left\{c/b\right\}^{\frac{2}{3}}$, the roots are in Geometrical Progression.

If $a^2 = b^2 + c^2$, we get $z = c^2/b^2$.

If $b = c$, we get $z = \left\{a^2 - 2b^2 \pm a(a^2-4b^2)^{\frac{1}{2}}\right\}/2b^2$.

If $a = b$, we get $z = \tfrac{1}{2} \pm (4c^2 - 3b^2)^{\frac{1}{2}}/2b$.

&c. &c.

8758. (Professor Ignacio Beyens.)—Résoudre le système d'équations

$$a^x(a^y + a^z + a^{x+y+z}) - b = 0, \quad a^y(a^x + a^z + a^{x+y+z}) - c = 0,$$
$$a^z(a^x + a^y + a^{x+y+z}) - d = 0.$$

Solution.

We have $a^z + a^{x+y+z} = b/a^x - a^y = c/a^y - a^x$ .....(1), (2),

and $b/a^x - a^z = d/a^z - a^x$ .....(1), (3),

also $c/a^y - a^z = d/a^z - a^y$ .....(2), (3),

therefore $a^y - a^x + c/a^y = a^z - a^x + d/a^x$.....(1), (2) and (1), (3),

and $b/a^x + a^x - a^y = a^z - a^y + d/a^z$ .....(1), (2) and (2), (3),

also $b/a^x + a^x - a^z = a^y - a^z + c/a^y$ .....(1), (3) and (2), (3).

Again $a^y + c/a = b/a^x + a^x = a^z + d/a^z$ from the last three equations,

$= a^z + a^{x+y+z}$ from the first equation;

therefore $a^{x+y+z} = d/a^z$; therefore $d = a^{x+y+2z}$.

Equating the two values of d, we easily establish $a^x + a^y = 0$; then $x = y$, then $a = 0$, whence $b = c = d = 0$.

8812. (R. W. D. Christie.)—If a_n be the coefficient of x^n in the development of $1/(1+2x-x^2)$, prove that

$$(2m+1)\pi+\tfrac{1}{4}\pi = 2\tan^{-1}\frac{a_{n+1}}{a_n}+(-1)^n\tan^{-1}\frac{1}{a_n^2+2a_na_{n+1}-a_{n+1}^2},$$

where m is a positive or negative integer or zero.

Solution.

The coefficient of x^n in the development of $1/(1+2x-x^2)$ is

$$(-1)^n\cdot\left\{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}\right\}/2\sqrt{2};$$

$$\therefore\ a_n/a_{n+1} = \left\{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}\right\}/\left\{(1+\sqrt{2})^{n+2}-(1-\sqrt{2})^{n+2}\right\}$$

$$= n^{\text{th}} \text{ convergent of } \sqrt{2} \text{ minus unity.}$$

Thus $2\tan^{-1}a_n/(a_{n+1})+\&c. = 2\tan^{-1}(\sqrt{2}-1) = 2\tan^{-1}(22\tfrac{1}{2}^\circ)$ &c. $= \tfrac{1}{4}\pi$.

It may be worth while to note that the original equation may be put in two forms; thus, suppose $n = 0$, 1 or 2, then we get briefly

$$2\tan^{-1}\tfrac{1}{2}-\tan^{-1}\tfrac{1}{7} \text{ or } 2\tan^{-1}\tfrac{1}{2}+\tan^{-1}\tfrac{1}{7} = \tfrac{1}{4}\pi,$$

$$2\tan^{-1}\tfrac{2}{3}+\tan^{-1}\tfrac{1}{17} \text{ or } 2\tan^{-1}\tfrac{2}{7}-\tan^{-1}\tfrac{1}{47} = \tfrac{1}{4}\pi;$$

or more fully thus,—

$$n = 2,\quad a_n = a_2 = 5,\quad a_{n+1} = a_3 = 12,$$

$$(2m+1)\pi+\tfrac{1}{4}\pi = 2\tan^{-1}(-\tfrac{1}{5})+\tan^{-1}\frac{1}{25-120-144}$$

$$= -2\tan^{-1}\tfrac{1}{5}-\tan^{-1}\tfrac{1}{239}$$

$$= \tan^{-1}\tfrac{120}{119}-\tan^{-1}\tfrac{1}{239} = \tan^{-1}1 = \text{Machin's Series.}$$

The second form is $2\tan^{-1}\frac{7}{17}+\tan^{-1}\frac{1}{239} = \tan^{-1}1$, &c., &c.

The coefficient of x^n in the development of $1/(1+2x-x^2)$ may also be found as the sum of $n+1$ terms of the following series:—

$$\left(2^n+\frac{n-1}{1}2^{n-2}+\frac{n-2\,.\,n-3}{2!}2^{n-4}+\frac{n-3\,.\,n-4\,.\,n-5}{3!}2^{n-6}+\&c.\right)(-1)^n.$$

The theorem may be extended thus:—

$$2\tan^{-1}\frac{a_n}{a_{n+1}}-2\tan^{-1}\frac{1}{a_{2n+3}}+\tan^{-1}\frac{1}{a_{n+3}+a_{n+4}} = \tan^{-1}1, \text{ \&c., \&c., } \textit{ad lib.}$$

Example: $\tfrac{1}{4}\pi = 2\tan^{-1}\frac{5}{18}-2\tan^{-1}\frac{1}{239}+\tan^{-1}\frac{1}{13395}$.

We have also immediately from the above

$$\tfrac{1}{4}\pi = \tan^{-1}\frac{a_n}{a_{n+2}}+\tan^{-1}\frac{a_{n+1}-a_n}{a_{n+1}+a_n}.$$

Examples: $\tfrac{1}{4}\pi = \tan^{-1}\tfrac{1}{2}+\tan^{-1}\tfrac{1}{3}$ (Euler's Series),

$\tfrac{1}{4}\pi = \tan^{-1}\tfrac{2}{5}+\tan^{-1}\tfrac{3}{7}$.

&c. &c.

8849. (R. W. D. Christie.)—Prove that Bernoulli's numbers are of the form $\dfrac{6N\pm1}{6n}$.

Solution.

These numbers consist of a fraction *plus* a whole number or zero.

The fraction is easily obtained. For, suppose we want B_{10}. Taking the divisors of its double (20), we have 1, 2, 4, 5, 10, 20. Now increase each of these by unity and take the primes, and we have 2, 3, 5, 11, and

fraction is $\frac{1}{2}-\frac{1}{3}-\frac{1}{5}-\frac{1}{11} = -\frac{41}{330}$, therefore $B_{10} = -\frac{174611}{330}$.

Similarly, if we want B_{11},

Divisors, 1, 2, 11, 22; Primes, 2, 3, 23; Fraction $\frac{1}{2}-\frac{1}{3}-\frac{1}{23}$;

therefore $B_{11} = \frac{17}{138}+6612 = \frac{912477}{138}$, and so on.

Now it is easy to see that, as the fraction is formed from an even number, it is always divisible by *unity* and *two*, which numbers increased by unity become 2, 3; therefore the *denominator* is always of form $6n$, and, as every prime above 3 is of form $6N \pm 1$, therefore the numerator is also of that form, as we easily see from the above examples, *e.g.*,

$$B_5 = \frac{1}{2}-\frac{1}{3}-\frac{1}{11} = \frac{1}{6}-\frac{1}{11} = \frac{5}{66},$$

$$B_6 = \frac{1}{2}-\frac{1}{3}-\frac{1}{5}-\frac{1}{7}-\frac{1}{13} = \frac{1}{6}-\frac{1}{5}-\frac{1}{7}-\frac{1}{13} = -\frac{691}{17710}, \text{ and so on.}$$

8877. (Professor NILKANTA SARKAR, M.A.)—Find (1) the interest with which any capital whatever would be increased at the end of a year, if the interest due at every instant (*i.e.*, second) of the year were itself to become capital and bear interest; and (2) what, if the increase of capital were regarded as constant.

Solution.

Let P = principal, m = no. of instants in 1 year, R = amount of £1 for 1 instant, r = interest of £1 for 1 year. Then amount of P for 1 year

is $$A = P\left(1+\frac{r}{m}\right)^m = PR^m,$$

therefore interest accumulated = $P(R^m-1)$ .....(1),

and, substituting A for P, we have $PR^m(R^m-1)$.....(2).

8889. (Rev. T. R. TERRY, M.A.)—A set of rectangular axes is brought from one given position to another by a rotation round the common origin. Show how to find the axis and angle of rotation in the case in which the second position relative to the first is given by the equations $4x' = x+y(1-2\sqrt{2})+z(2+\sqrt{2})$,

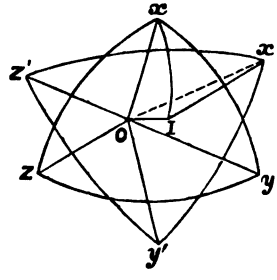
$4y' = x(1+2\sqrt{2})+y+z(\sqrt{2}-2)$, $4z' = x(\sqrt{2}-2)+y(\sqrt{2}+2)+2z$.

Solution by the PROPOSER.

To find ON the axis of rotation, put $x' = x$, $y' = y$, $z' = z$, and solve, thus

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{\sqrt{2}},$$

therefore direction cosines of ON are $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\sqrt{2}$. To find θ the angle of rotation. If any point P comes to Q, and PN, QN be the perpendiculars on ON, then $PQ = 2PN \sin \frac{1}{2}\theta$. Let the coordinates of P be (1, 0, 0). Then coordinates of Q are



$$\left\{\tfrac{1}{4},\ \tfrac{1}{4}(1+2\sqrt{2}),\ \tfrac{1}{4}(\sqrt{2}-2)\right\},$$

therefore $PQ = \frac{1}{4}\sqrt{6}$. And $ON = \frac{1}{2}$, therefore

$$PN = \tfrac{1}{4}\sqrt{3}, \text{ therefore } \sin\tfrac{1}{2}\theta = \frac{1}{\sqrt{2}}, \text{ therefore } \theta = 90°.$$

The rotation is therefore a rotation through a right angle, round the line $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\sqrt{2}$.

8997. (Professor HANUMANTA RAU, B.A.)—Solve the equation

$$x^5 + 5ax^4 + 20x^3 + 20ax^2 + 20x + 4a = 0.$$

Solution.

Let $f(x) = 0$ be the given equation. Then

$$f'(x) = x^4 + 4ax^3 + 12x^2 + 8ax + 4.$$

The G. C. M. of $f(x)$ and $f'(x)$ will give us equal roots, therefore we must have $a = \pm\sqrt{2}$, therefore

$$f'(x) = x^4 \pm 4\sqrt{2}\,x^3 + 12x^2 \pm 8\sqrt{2}\,x + 4 = 0.$$

Now, if $r^2 - p^2 s = 0$, any biquadratic may be solved as a quadratic, therefore $\{x^2 \pm 2\sqrt{2}\,x + 2\}^2 = 0$, therefore $x \mp \sqrt{2} = 0$ (the opposite sign to that of a must be taken).

9091. (R. W. D. CHRISTIE.)—Arrange the first sixteen natural numbers in the form of a determinant of the fourth order or degree, the sum of whose rows, columns, and diagonals are all equal.

Solution.

Any n^2 natural numbers may be arranged in the form of a determinant by making the principal diagonals consist of two arithmetical pro-

gressions as follows:—

1	$n^2-\frac{1}{2}n+1$	$n^2-\frac{1}{2}n$	n
$n(\frac{1}{2}n+1)$	$n+2$	$2n-1$	$\frac{1}{2}n^2+1$
$n(\frac{1}{2}n)$	$3n-2$	$2n+3$	$\frac{1}{2}n^2-n+1$
$4n-3$	$\frac{1}{2}n+1$	$\frac{1}{2}n$	n^2 or $3n+4$

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

If $n = 4$ we have the sum $s = \{n(n^2+1)\}/2 = 34$.

The remaining squares are easily filled up from the remaining numbers, since they work in pairs whose sum $= (n^2+1)$—thus 15 and 2, 14 and 3, 9 and 8. Similarly for any even square. [See Question 9278, on p. 203.]

9223. (H. L. Orchard, B.Sc., M.A.)—Two crews, rowing the one 18 miles with the current, and the other 21 miles against it, pass one another, and complete their courses by rowing for 1 and 5 hours more, respectively. Supposing the crews to start at the same time and to row uniformly, find their rates of speed.

Solution.

This question is ambiguously worded. A simple solution is—
Let x and y = rates of speed. Then we have

$$21-x = 5y, \quad (18-x)/x = 21/y-5.$$

Whence $x = 6$, $y = 3$ miles per hour.

[*Otherwise*:—Let the one crew, rowing against the stream, meet the other, rowing with the stream, in t hours; and in this time let the former have gone x miles, and therefore the latter $18-x$ miles; then, since they each row uniformly, we have at once $t : 5 = x : 21-x$, and $t : 1 = 18-x : x$; whence $t = 2$, and $x = 6$; thus the rates are 3 miles and 6 miles an hour, respectively.]

9225. (R. W. D. Christie.)—Show that the sum of the primes $= 30\Sigma(\Sigma+1)+18-C$; where Σ represents the number of the set (of four) to which the given prime belongs, and C = sum of multiples of primes. [*Ex. gr.*, To find the sum of primes to 97 inclusive,

$$30\Sigma(\Sigma+1) = 30\times 6\times 7 = 1260, \quad C = 7(7+11+13) = 217;$$

hence the required sum is $1260+18-217 = 1061$.]

Solution.

Primes >7 end in the digits 1, 3, 7, or 9. Distributing the primes *with their multiples* into groups of four, we find that they recur in the following order of their last digits, viz., (1, 3, 7, 9), (3, 9, 1, 7), and that the sums of these groups = 60N, where N = 1, 2, 3, &c., and therefore the sum = $30\Sigma(\Sigma+1)$. Examples—

(11, 13, 17, 19), (23, 29, 31, 37), (41, 43, 47, [49]), (53, 59, 61, 67).

We find the number of the group from its last member; thus, if we call the groups A and B respectively, we find the number

$$\Sigma_A = \tfrac{1}{3}(2T+1), \quad \Sigma_B = \tfrac{2}{3}T,$$

where T represents the value of the tens in the last member of the group.

To complete the rule, we add the first five primes (two of which are in many respects exceptional), viz. 18, and subtract C, the sum of the multiples of primes which are occasionally required to complete the groups.

Ex. gr., Sum the primes to 199 inclusive. The group is

(191, 193, 197, 199), therefore $\Sigma_A = \tfrac{1}{3}(2T+1) = \tfrac{1}{3}(2\times 19+1) = 13$,

therefore $30\Sigma(\Sigma+1)+18-C = 30\times 13\times 14+18-1250 = 4228$.

N.B.—There may be a little trouble at first in fixing the group to which a given prime belongs, but a little practice will obviate this. Ex.: To which group does 193 belong? A or B group? (191, 193, 197, 199) or (193, 199, 201, 207)? Here T = 19 or 20. The former gives the only integral result, viz., $\tfrac{1}{3}(2T+1) = 13$. Hence it is the right group. In adding, allowance will of course be made for a prime placed in the middle of a group.

9278. (R. W. D. Christie.)—Show that it is always possible to arrange the first (n^2) natural numbers in the form of a determinant the sum of whose rows, columns, and diagonals is $\frac{1}{2}n(n^2+1)$ when $n>2$.

Example.

1,	20,	33,	34,	17,	6
32,	8,	30,	7,	11,	23
24,	35,	15,	16,	2,	19
18,	10,	21,	22,	27,	13
5,	26,	9,	28,	29,	14
31,	12,	3,	4,	25,	36

Solution.

[See Quest. 9091, on pp. 201, 202.]

We have $1+(n+2)+\&c. = n+(2n-1)+\&c. = \{n(n^2+1)\}/2$.

N.B.—The squares may be arranged in various ways. The *odd* squares are mechanically figured thus:—

(1)

1,	20,	16,	23,	5
15,	7,	12,	9,	22
24,	18,	13,	8,	2
4,	17,	14,	19,	11
21,	3,	10,	6,	25

(2)

17,	24,	1,	8,	15
23,	5,	7,	14,	16
4,	6,	13,	20,	22
10,	12,	19,	21,	3
11,	18,	25,	2,	9

(1) is arranged with each diagonal as an arithmetical progression.

(2) Any *odd* square may be mechanically arranged like this one.

Begin in the top central square, and follow the Nos. 1, 2, 3, &c., proceeding diagonally. If any square be occupied, place underneath and go on as before. When the edge of the square is reached, go to the extreme square of neighbouring row or column. One diagonal always forms a sequence.

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